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Partial Differential Equations I

Foundations of the Classical Theory



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Linear Partial Differential Equations. Foundations of the Classical Theory

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