

D.F. Parker

Fields, Flows and Waves

An Introduction to Continuum Models

With 90 Figures



Springer

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