Error-Correcting Codes and Finite Fields

Student edition

PART 1 BASIC CODING THEORY

1 Introduction

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Errors of transmission. Examples from natural language. Channel models. The binary symmetric channel. Three simple codes (a parity check code, a triple repetition code, and a triple parity check code).

2 Block codes, weight, and distance

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Block codes. Block length, message block length, and rate. Definition of Hamming weight and distance. Minimum distance, error detection, and error correction. Block and message success probabilities. Calculation of error detection/correction probabilities for the examples of Chapter 1. Discussion of Shannon's theorem (without proof).

3 Linear codes

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Definition of linear codes and fields. Dimension and rate. The generator matrix. Standard form generator matrices and systematic encoding, Message and check bits. The check matrix. Uniqueness of standard form generator and check matrices.

4 Error processing for linear codes

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Decoding by cosets (standard array). Coset leaders and syndromes. Code can correct single errors if and only if check matrix has distinct non-zero columns. Conditions for multiple error correction.

5 Hamming codes and the binary Golay codes

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Definition of the sequence of binary Hamming codes Ham(k) by their check matrices. Success probabilities for Hamming codes. Long Hamming codes are very efficient, but poor at correcting errors. Perfect codes. Construction of the binary Golay codes by Turyn's method.

Appendix LA Linear algebra

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The laws of arithmetic, rings, domains, and fields. Elementary vector space theory. Bases and dimension. Elementary matrix theory. Row operations, rank, and nullity. Vandermonde matrices.

PART 2 FINITE FIELDS

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The need for fields other than Z/2. An attempt to construct a field of order 16. Z/16 will not do. Polynomial arithmetic. Table of GF(16).

7 Euclid's algorithm

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Division with remainder. Euclidean domains with F[x] and Z as examples. Euclid's algorithm in tabular form for a Euclidean domain. Finding the highest common factor in the form (a, b) = ua + vb.

Extras. Relations between entries in the table for Euclid's algorithm. Continued fractions, Convergents, the entries in the tabular form of Euclid's algorithm and convergents to continued fractions.

8 Invertible and irreducible elements

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Definition of invertible elements in a Euclidean domain. Definition of irreducible elements in a Euclidean domain. The 1-trick and the key property of irreducible elements. Discussion of unique factorization.

Extras. Proof of unique factorization.

9 The construction of fields

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Construction of the factor ring (residue class ring) D/a. D/a is a field if and only if a is irreducible. Using Euclid's algorithm to perform field arithmetic in $F\{x\}//(x)$. Examples: GF(16) as $GF(2)[x]/(x^4 + x^2 + 1)$, Z/787.

10 The structure of finite fields

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The prime field and the characteristic. The order of a finite field. The Frobenius automorphism $x \to x^p$. Fermat's little theorem: if F has order q then all its elements are roots of $x^q - x$. Example: GF(16).

11 Roots of polynomials

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The evaluation map. Its basic properties (i.e. it is a homomorphism). The formal derivative. Horner's scheme for evaluating a polynomial. Extension of Horner's scheme to evaluate the derivative. Multiple roots. The minimal polynomial of α . Characterization of the minimal polynomial and the set (ideal) of polynomials with α as a root. List of minimal polynomials of elements of GF(16). Isomorphism $F[\alpha] \cong F[x]/mp_{\alpha}(x)$. Construction of a field containing a root of a given polynomial. Existence of finite fields of all legal orders.

Extras. Calculation of the minimum polynomial of β using the Frobenius automorphism.

12 Primitive elements

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Definition of primitive elements. Primitive elements of GF(16). Logarithms for calculating products and quotients in finite fields. Zech logarithms for calculating sums. Primitive polynomials. Existence of primitive elements. Existence of subfields of all legal orders. Isomorphism of fields of the same order. The polynomials $x^a - x$ is the product of all irreducible polynomials of degree dividing a.

Extras. The number of irreducible polynomials of a given degree.

Appendix PF Polynomials over a field

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Recapitulation of the basic theory of polynomials over a field. Definition, addition, multiplication, degree, F[x] is an integral domain. Division with remainder. Polynomials in two indeterminates.

PART 3 BCH CODES AND OTHER POLYNOMIAL CODES

13 BCH codes as subcodes of Hamming codes

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Example: BCH(4, 2) constructed from Ham(4) by extending the check matrix H_a . Extensions must not be linear (or quadratic). View H_k as having entries in GF(2*). Criterion for multiple error correction. Vandermonde matrices. The full check matrix $V_{4,2}$ and the reduced check matrix $H_{4,2}$ (V_k , and $H_{k,t}$ in general). Example BCH(4, 3). BCH(k, t) can correct t errors per block. It has block length $2^k - 1$ and dimension $\ge 2^k - 1 - kt$.

14 BCH codes as polynomial codes

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15 BCH error correction: (1) the fundamental equation

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17 Reed-Solomon codes and burst error correction

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