LINEAR ALGEBRA

GEORGI E. SHILOV

Professor of Mathematics
Moscow University

Revised English Edition
Translated and Edited by

Richard A. Silverman

DOVER PUBLICATIONS, INC., NEW YORK

CONTENTS

chapter 1

DETERMINANTS 1

1.1. Number Fields 1
1.2. Problems of the Theory of Systems of Linear Equations 3
1.3. Determinants of Order n 5
1.4. Properties of Determinants 8
1.5. Cofactors and Minors 12
1.6. Practical Evaluation of Determinants 16
1.7. Cramer's Rule 18
1.8. Minors of Arbitrary Order. Laplace's Theorem 20
1.9. Linear Dependence between Columns 23
Problems 28

chapter 2

LINEAR SPACES 31

2.1. Definitions 31
2.2. Linear Dependence 36
2.3. Bases, Components, Dimension 38
2.4. Subspaces 42
2.5. Linear Manifolds 49
2.6. Hyperplanes 51
2.7. Morphisms of Linear Spaces 53
Problems 56

chapter 3

SYSTEMS OF LINEAR EQUATIONS 58

3.1. More on the Rank of a Matrix 58
3.2. Nontrivial Compatibility of a Homogeneous Linear System 60
3.3. The Compatibility Condition for a General Linear System 61
3.4. The General Solution of a Linear System 63
3.5. Geometric Properties of the Solution Space 65
3.6. Methods for Calculating the Rank of a Matrix 67
Problems 71

chapter 4

LINEAR FUNCTIONS OF A VECTOR ARGUMENT 75

4.1. Linear Forms 75
4.2. Linear Operators 77
4.3. Sums and Products of Linear Operators 82
4.4. Corresponding Operations on Matrices 84
4.5. Further Properties of Matrix Multiplication 88
4.6. The Range and Null Space of a Linear Operator 93
4.7. Linear Operators Mapping a Space $\mathbf{K}_n$ into Itself 98
4.8. Invariant Subspaces 106
4.9. Eigenvectors and Eigenvalues 108
Problems 113

chapter 5

COORDINATE TRANSFORMATIONS 118

5.1. Transformation to a New Basis 118
5.2. Consecutive Transformations 120
5.3. Transformation of the Components of a Vector 121
5.4. Transformation of the Coefficients of a Linear Form 123
5.5. Transformation of the Matrix of a Linear Operator 124
*5.6. Tensors 126
Problems 131

chapter 6

THE CANONICAL FORM OF THE MATRIX OF A LINEAR OPERATOR 133

6.1. Canonical Form of the Matrix of a Nilpotent Operator 133
6.2. Algebras. The Algebra of Polynomials 136
6.3. Canonical Form of the Matrix of an Arbitrary Operator 142
6.4. Elementary Divisors 147
6.5. Further Implications 153
6.6. The Real Jordan Canonical Form 155
*6.7. Spectra, Jets and Polynomials 160
*6.8. Operator Functions and Their Matrices 169
Problems 176

chapter 7

BILINEAR AND QUADRATIC FORMS 179

7.1. Bilinear Forms 179
7.2. Quadratic Forms 183
7.3. Reduction of a Quadratic Form to Canonical Form 185
7.4. The Canonical Basis of a Bilinear Form 190
7.5. Construction of a Canonical Basis by Jacobi's Method 192
7.6. Adjoint Linear Operators 196
7.7. Isomorphism of Spaces Equipped with a Bilinear Form 199
*7.8. Multilinear Forms 202
7.9. Bilinear and Quadratic Forms in a Real Space 204
Problems 210

chapter 8

EUCLIDEAN SPACES 214

8.1. Introduction 214
8.2. Definition of a Euclidean Space 215
8.3. Basic Metric Concepts 216
8.4. Orthogonal Bases 222
8.5. Perpendiculars 223
8.6. The Orthogonalization Theorem 226
8.7. The Gram Determinant 230
8.8. Incompatible Systems and the Method of Least Squares 234
8.9. Adjoint Operators and Isometry 237
Problems 241

chapter 9

UNITARY SPACES 247

9.1. Hermitian Forms 247
9.2. The Scalar Product in a Complex Space 254
9.3. Normal Operators 259
9.4. Applications to Operator Theory in Euclidean Space 263
Problems 271

chapter 10

QUADRATIC FORMS IN EUCLIDEAN AND UNITARY SPACES 273

10.1. Basic Theorem on Quadratic Forms in a Euclidean Space 273
10.2. Extremal Properties of a Quadratic Form 276
10.3. Simultaneous Reduction of Two Quadratic Forms 283
10.4. Reduction of the General Equation of a Quadric Surface 287
10.5. Geometric Properties of a Quadric Surface 289
*10.6. Analysis of a Quadric Surface from Its General Equation 300
10.7. Hermitian Quadratic Forms 308
Problems 310

*chapter 11

FINITE-DIMENSIONAL ALGEBRAS AND THEIR REPRESENTATIONS 312

11.1. More on Algebras 312
11.2. Representations of Abstract Algebras 313
11.3. Irreducible Representations and Schur's Lemma 314
11.4. Basic Types of Finite-Dimensional Algebras 315
11.5. The Left Regular Representation of a Simple Algebra 318
11.6. Structure of Simple Algebras 320
11.7. Structure of Semisimple Algebras 323
11.8. Representations of Simple and Semisimple Algebras 327
11.9. Some Further Results 331
Problems 332

*Appendix

CATEGORIES OF FINITE-DIMENSIONAL SPACES 335

A.1. Introduction 335
A.2. The Case of Complete Algebras 338
A.3. The Case of One-Dimensional Algebras 340
A.4. The Case of Simple Algebras 345
A.5. The Case of Complete Algebras of Diagonal Matrices 353
A.6. Categories and Direct Sums 357

HINTS AND ANSWERS 361

BIBLIOGRAPHY 379

INDEX 381