

Numerical Computation
of
INTERNAL AND EXTERNAL
FLOWS

Volume 1: Fundamentals of Numerical Discretization

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