# MATHEMATICAL RECREATIONS AND ESSAYS

W.W. ROUSE BALL AND H.S.M. COXETER

THIRTEENTH EDITION

DOVER PUBLICATIONS, INC. NEW YORK

## CONTENTS

1 ARITHMETICAL RECREATIONS 3
To find a number selected by someone 5
Prediction of the result of certain operations 8
Problems involving two numbers 11
Problems depending on the scale of notation 12
Other problems with numbers in the denary scale 14
Four fours problem 16
Problems with a series of numbered things 17
Arithmetical restorations 20
Calendar problems 26
Medieval problems in arithmetic 27
The Josephus problem. Decimation 32
Nim and similar games 36
Moore's game 38
Kayles 39
Wythoff's game 39
Addendum on solutions 40
II ARITHMETICAL RECREATIONS (continued) 41
Arithmetical fallacies 41
Paradoxical problems 44
Probability problems 45
Permutation problems 48
Bachet's weights problem 50
The decimal expression for 1/n 53
Decimals and continued fractions 54

### - XII CONTENTS

Rational right-angled triangles 57
Triangular and pyramidal numbers 59
Divisibility 60
The prime number theorem 62
Mersenne numbers 64
Perfect numbers 66
Fermat numbers 67
Fermat's Last Theorem 69
Galois fields 73

Geometrical fallacies 76
Geometrical fallacies 76
Geometrical paradoxes 84
Continued fractions and lattice points 86
Geometrical dissections 87
Cyclotomy 94
Compass problems 96
The five-disc problem 97
Lebesgue's minimal problem 99
Kakeya's minimal problem 99
Addendum on a solution 102

IV GEOMETRICAL RECREATIONS (continued) Statical games of position 103 Three-in-a-row. Extension to p-in-a-row 103 Tessellation 105 Anallagmatic pavements 107 Polyominoes 109 Colour-cube problem 113 Squaring the square 115 Dynamical games of position 116 Shunting problems 116 Ferry-boat problems Geodesic problems Problems with counters or pawns 121 Paradromic rings 127 Addendum on solutions

POLYHEDRA 130 Symmetry and symmetries 130 The five Platonic solids 131 133 Kepler's mysticism Pappus, on the distribution of vertices 134 Compounds 135 The Archimedean solids Mrs. Stott's construction Equilateral zonohedra 141 The Kepler-Poinsot polyhedra 144 The 59 icosahedra 146 Solid tessellations Ball-piling or close-packing The sand by the sea-shore 151 Regular sponges 152 Rotating rings of tetrahedra 154 The kaleidoscope

162 VI CHESS-BOARD RECREATIONS Relative value of pieces The eight queens problem 166 Maximum pieces problem 172 Minimum pieces problem 172 Re-entrant paths on a chess-board 175 175 Knight's re-entrant path 186 King's re-entrant path Rook's re-entrant path 187 Bishop's re-entrant path Routes on a chess-board Guarini's problem 189 Latin squares 189 Eulerian squares 190 Euler's officers problem Eulerian cubes 192

VII MAGIC SQUARES 193 Magic squares of an odd order 195

#### xiv CONTENTS

Magic squares of a singly-even order 196 Magic squares of a doubly-even order 199 Bordered squares 200 Number of squares of a given order 201 Symmetrical and pandiagonal squares 202 Generalization of De la Loubère's rule 204 Arnoux's method 206 Margossian's method 207 Magic squares of non-consecutive numbers 210 Magic squares of primes 211 Doubly-magic and trebly-magic squares 212 Other magic problems 213 Magic domino squares 213 Cubic and octahedral dice 214 Interlocked hexagons 215 Magic cubes 216

VIII MAP-COLOURING PROBLEMS 222
The four-colour conjecture 222
The Petersen graph 225
Reduction to a standard map 227
Minimum number of districts for possible failure 230
Equivalent problem in the theory of numbers 231
Unbounded surfaces 232
Dual maps 234
Maps on various surfaces 234
Pits, peaks, and passes 238
Colouring the icosahedron 238

IX UNICURSAL PROBLEMS 243
Euler's problem 243
Number of ways of describing a unicursal figure 250
Mazes 254
Trees 260
The Hamiltonian game 262
Dragon designs 266

X COMBINATORIAL DESIGNS 271 A projective plane 271 Incidence matrices 272 An Hadamard matrix 273 An error-correcting code 274 A block design 276 Steiner triple systems 278 Finite geometries 281 Kirkman's school-girl problem 287 Latin squares 290 The cube and the simplex 295 Hadamard matrices 296 297 Picture transmission Equiangular lines in 3-space 299 Lines in higher-dimensional space 303 C-matrices 308 Projective planes 310

The fifteen puzzle 312
The Tower of Hanoï 316
Chinese rings 318
Problems connected with a pack of cards 322
Shuffling a pack 323
Arrangements by rows and columns 325
Bachet's problem with pairs of cards 326
Gergonne's pile problem 328
The window reader 333
The mouse trap. Treize 336

THREE CLASSICAL GEOMETRICAL PROBLEMS 338

The duplication of the cube 339

Solutions by Hippocrates, Archytas, Plato, Menaechmus, Apollonius, and Diocles 341

Solutions by Vieta, Descartes, Gregory of St. Vincent, and Newton 343

#### xvi CONTENTS

The trisection of an angle 344
Solutions by Pappus, Descartes, Newton, Clairaut, and
Chasles 334

The quadrature of the circle 347

. Origin of symbol m 349

 Geometrical methods of approximation to the numerical value of π 349

Results of Egyptians, Babylonians, Jews 350
Results of Archimedes and other Greek writers 351
Results of European writers, 1200–1630 352
Theorems of Wallis and Brouncker 355
Results of European writers, 1699–1873 356
Approximations by the theory of probability 359

XIII CALCULATING PRODIGIES 360 John Wallis, 1616-1703 361 Buxton, circ. 1707-1772 361 Fuller, 1710-1790; Ampère 364 Gauss, Whately 365 Colburn, 1804-1840 365 Bidder, 1806-1878 367 Mondeux, Mangiamele 372 Dase, 1824-1861 372 Safford 1836-1901 374 Zamebone, Diamandi, Rückle 375 Inaudi, 1867- 375 Types of memory of numbers 377 Bidder's analysis of methods used 378 Multiplication 379 Digital method for division and factors 381 Square roots. Higher roots 382 Compound interest 384 Logarithms 385 Alexander Craig Aitken 386

XIV CRYPTOGRAPHY AND CRYPTANALYSIS 388 Cryptographic systems 389 Transposition systems 391
Columnar transposition 392
Digraphs and trigraphs 394
Comparison of several messages 397
The grille 401
Substitution systems 402
Tables of frequency 404
Polyalphabetic systems 406
The Vigenère square 407
The Playfair cipher 410
Code 412
Determination of cryptographic system 414
A few final remarks 416
Addendum: References for further study 418

INDEX 419