HANDBOOK OF APPLICABLE MATHEMATICS

Chief Editor: Walter Ledermann

Volume I: Algebra

Edited by

Walter Ledermann

and

Steven Vajda

University of Sussex

A Wiley-Interscience Publication

JOHN WILEY & SONS

Chichester - New York - Brisbane - Toronto

Contents

INTRODUCTION xv

1. SETS 1
1.1. Sets 1
1.2. Set algebra 3
1.2.1. Unions and intersections 3
1.2.2. Venn diagrams 5
1.2.3. Subsets 7
1.2.4. Complements 9
1.2.5. Membership tables 10
1.2.6. Ordered pairs 11
1.2.7. Sets of sets 13
1.3. Relations 15
1.3.1. The definition of a relation 15
1.3.2. Order relations 16
1.3.3. Equivalence relations 16
1.3.4. The integers modulo *n* 18
1.4. Functions 20
1.4.1. The definition of a function 20
1.4.2. Fundamental properties of functions 24
1.5. Binary operations 27
1.6. Cardinal numbers 29

2. NUMBERS 31
2.1. The natural numbers 31
2.2. The integers 35
2.2.1. Definition and properties of the integers 35
2.2.2. Further properties of integers 38
2.3. Rings 40
2.4. Rational numbers and fields 42
2.4.1. Definition and properties of the rational numbers . . . 42
2.4.2. Fields 44

2.5. Decimal expansions 47
2.6. Real numbers 52
2.6.1. Definition of the real numbers 52
2.6.2. Properties of real numbers 54
2.6.3. Sets of real numbers 56
2.6.4. Roots of real numbers 59
2.7. Complex numbers 60
2.7.1. The field of complex numbers 60
2.7.2. Geometrical representation of complex numbers . . 64
2.7.3. Polar coordinates of complex numbers 70
2.7.4. Roots of complex numbers 73
2.8. Quaternions 75

3. ARITHMETIC 79
3.1. Powers 79
3.2. Inverses 81
3.3. Roots 82
3.4. Definition of a^b for real $a > 0$ and any real b . . . 85
3.5. The function $f(x) = a^x$ for $a > 0$ 86
3.6. Logarithms 87
3.7. Permutations 90
3.8. Combinations 92
3.9. Partitions and more general computations 93
3.10. Generating functions and relations between binomial coefficients . 95
3.11. Use of generating functions in enumeration problems . . 97

4. NUMBER THEORY 101
4.1. Divisibility 101
4.1.1. Division in rings 101
4.1.2. Euclidean domains 102
4.1.3. Common divisors 104
4.2. Primes and prime decomposition 107
4.3. Congruences 110
4.3.1. The algebra of congruences 110
4.3.2. Finite rings 113
4.3.3. Finite fields 115

5. LINEAR ALGEBRA 117
5.1. Displacements 117
5.2. Vector spaces 119
5.3. Linear dependence 121
5.4. Basis and dimension 125
5.5. Subspaces 127
5.6. Rank 130
5.7. Linear equations—preliminaries 133

5.8. *Linear equations—the regular case* 134
5.9. *Linear equations—homogeneous systems* 136
5.10. *Linear equations—non-homogeneous systems* 137
5.11. *Linear maps* 141
5.12. *Change of basis* 146
5.13. *Linear transformations* 150

6. *MATRICES AND DETERMINANTS* 153
6.1. *Motivation* 153
6.2. *Matrix algebra* 155
6.3. *Formal laws* 160
6.4. *The inverse matrix* 160
6.5. *Transposition* 167
6.6. *Partitioned matrices* 168
6.7. *Matrices of special types* 170
6.8. *Motivation for determinants* 177
6.9. *Formal definition of a determinant* 180
6.10. *General laws for determinants* 182
6.11. *Cofactors* 184
6.12. *Cramer's Rule* 187
6.13. *Minors* 189
6.14. *Some special determinants* 191
6.15. *The Kronecker product* 193

7. *EIGENVALUES* 195
7.1. *Geometrical motivation* 195
7.2. *Simple algebraic properties* 196
7.3. *The characteristic polynomial* 198
7.4. *Similar matrices* 203
7.5. *Matrix equations* 206
7.6. *The Jordan canonical form* 208
7.7. *Unitary (orthogonal) equivalence* 215
7.8. *The Principal Axes theorem* 216
7.9. *Extremal properties* 222
7.10. *Spectral decomposition* 225
7.11. *Non-negative matrices* 227
7.12. *Simultaneous diagonalization* 230
7.13. *Inequalities for eigenvalues* 232

8. *GROUPS* 237
Introduction 237
8.1. *Permutations* 238
8.1.1. The algebra of permutations 238
8.1.2. Properties of permutations 242
8.1.3. Even and odd permutations 243

8.2. Groups 245
8.2.1. The group concept 245
8.2.2. Group tables 249
8.2.3. Generators and relations 250
8.2.4. Subgroups and cosets 255
8.2.5. Properties of group elements 258
8.2.6. Normal subgroups and factor groups 259
8.2.7. Homomorphisms between groups 262
8.2.8. Permutation groups and Cayley's theorem 265
8.2.9. Direct products 266
8.3. Representations of groups 268
8.3.1. Introduction 268
8.3.2. Matrix representations of groups 269
8.3.3. Reducibility of matrix representations 273
8.3.4. Linear representations of finite Abelian groups . . . 276
8.3.5. Group characters 278
8.3.6. Orthogonality properties of group characters . . . 281
8.3.7. Matrix representations of permutation groups . . . 283
8.4. Other mathematical structures 286

9. BILINEAR AND QUADRATIC FORMS 287
9.1. Classification of quadratic forms under congruence . . . 287
9.2. Definiteness of quadratic forms 293
9.3. Hermitian forms 296

10. METRIC VECTOR SPACES 299
10.1. Inner products and metrics 299
10.2. Inner product spaces 302
10.3. Complex inner product spaces 306

11. LINEAR PROGRAMMING 309
11.1. Introduction 309
11.1.1. The diet problem 309
11.1.2. Nomenclature 311
11.2. Primal and dual programmes 311
11.2.1. Dual of the diet problem 311
11.2.2. Matrix formulation of primal and dual programmes . . 312
11.2.3. Relations between primal and dual programmes . . 313
11.2.4. Results of duality theory 314
11.2.5. Alternative form for primal and dual programmes . . 315
11.3. The transportation problem 316
11.3.1. Formulation and solution of the transportation problem . 316
11.3.2. The assignment problem 321
11.3.3. The transshipment problem 321

11.4. The simplex method 324
11.4.1. Outline of simplex method 324
11.4.2. Determination of basic feasible solutions . . . 324
11.4.3. Choice of pivot 326
11.4.4. Operation of pivot rule 327
11.4.5. Solution of dual programme 329
11.4.6. Inverse matrix (revised simplex) method . . . 330
11.4.7. Dual simplex method 331
11.4.8. Degeneracy 331
11.4.9. Sensitivity analysis 331
11.4.10. Parametric linear programming 332
11.4.11. Decomposition method 333
11.5. Applications 333
11.5.1. A production scheduling problem 333
11.5.2. A cutting problem 334
11.5.3. Revenue maximization 335

12. INTEGER PROGRAMMING 339
Introduction 339
12.1. The applicability of integer programming 341
12.1.1. Problems involving discrete activities 341
12.1.2. Problems involving discrete resources 342
12.1.3. Fixed charge problems 342
12.1.4. Problems with logical conditions 342
12.1.5. Non-linear problems 343
12.1.6. Combinatorial problems 343
12.2. Building integer programming models 344
12.2.1. The uses of integer variables 345
12.2.2. The uses of integer constraints 345
12.2.3. The relationship between integer programming models and linear programming models 356
12.3. Solving integer programming models 359
12.3.1. General methods of solving integer programming models . 361
(a) The Branch and Bound algorithm 361
(b) The Cutting Planes algorithm 365
12.3.2. Specialist methods of solving integer programming models . 371
(a) Implicit enumeration 371
(b) The pseudo-Boolean algorithm 374
12.3.3. Other approaches to solving integer programming models . 377
(a) Partitioning a MIP model (Benders' decomposition) . 377
(b) Dynamic programming 377
(c) Group theory 379
(d) Lagrangean relaxation 380
(e) Approximate methods 383
12.4. Areas for future development 385

12.5. References and further reading 386
12.6. Bibliography 388

13. THE THEORY OF GAMES 391
13.1. Introduction 391
13.1.1. Background 391
13.1.2. Concepts and definitions 392
13.2. Matrix games 393
13.2.1. Pure strategies 393
13.2.2. Mixed strategies 396
13.2.3. Solutions and equilibrium points 397
13.2.4. Methods for solving small rectangular games . . 399
13.2.5. Dominance 401
13.2.6. Games against nature 403
13.2.7. Fixed points 405
13.3. More-person games 407
13.3.1. Non-cooperative games 407
13.3.2. Cooperative games 408
13.3.3. von Neumann-Morgenstern solution . . . 409
13.3.4. Another concept of solution: the Shapley value . 410
13.4. Non-zero-sum games 411
13.5. Extensive games 413
13.6. Infinite games 416
13.6.1. Pure strategies 417
13.6.2. Duels 419
13.6.3. Statistical decision procedures. 420

14. POLYNOMIALS AND RATIONAL FUNCTIONS . . . 423
14.1. Polynomials in a single variable 423
14.2. Algebra of polynomials 425
14.3. The division algorithm 425
14.4. Factorization 430
14.5. Roots of polynomials 431
14.6. Repeated factors 434
14.7. Locating the real zeros 435
14.8. Finding the number of real roots of a polynomial . . 436
14.9. Real rational functions 439
14.10. Partial fractions. 440
14.11. Applications of partial fractions 444
14.12. Linear recursive sequences 446
14.13. Solution of recurrence relations 447
14.14. Systems of recurrence relations 454
14.15. Polynomials in several indeterminates 456
14.16. Symmetric polynomials 458

15. FINANCIAL MATHEMATICS 463
15.1. Investments, annuities and loans 463
15.1.1. Interest 463
15.1.2. Annuities 464
15.1.3. Repayment of loans 465
15.1.4. Intrinsic rate of interest 467
15.2. Actuarial mathematics 468
15.2.1. Life contingencies 468
15.2.2. Policy values 470
15.2.3. Other assurance types 473
15.2.4. Continuous mortality 473

16. BOOLEAN ALGEBRA 477
16.1. Laws of a Boolean algebra 477
16.2. The inclusion relation 480
16.3. Truth tables 482
16.4. Switching circuits 484
16.5. Mathematical logic 488

17. UNITS OF MEASUREMENT 493
17.1. Introduction 493
17.1.1. Physical quantities, units and dimensions 493
17.1.2. Systems of units 494
17.2. SI units—Le Système International des Unités 495
17.2.1. The SI base units 496
17.2.2. SI supplementary units 497
17.2.3. SI derived units 498
17.2.4. SI prefixes 498
17.2.5. Rules for printing of SI symbols 499
17.2.6. Units outside the SI system 500
17.3. Other systems of units 501
17.4. Checking by dimensions 501
17.5. Conversion of units 502
17.6. Formulation of equations with inconsistent units 502
17.7. Tables of conversion factors between SI and non-SI units 503
17.8. Dimensional analysis 503
17.8.1. Method of indices 510
17.8.2. Method of dimensionless groups 511

INDEX 515