# Ordinary Differential Equations



## Ordinary Differential Equations



George F. Carrier
Harvard University

Carl E. Pearson University of Washington



Books in the Classics in Applied Mathematics series are monographs and textbooks declared out of print by their original publishers, though they are of continued importance and interest to the mathematical community. SIAM publishes this series to ensure that the information presented in these texts is not lost to today's students and researchers.

#### Editor-in-Chief

Robert E. O'Malley, Jr., University of Washington

#### **Editorial Board**

John Boyd, University of Michigan

Leah Edelstein-Keshet, University of British Columbia

William G. Faris, University of Arizona

Nicholas J. Higham, University of Manchester

Peter Hoff, University of Washington

Mark Kot, University of Washington

Hilary Ockendon, University of Oxford

Peter Olver, University of Minnesota

Philip Protter, Cornell University

Gerhard Wanner, L'Université de Genève

#### Classics in Applied Mathematics

C. C. Lin and L. A. Segel, Mathematics Applied to Deterministic Problems in the Natural Sciences

Johan G. F. Belinfante and Bernard Kolman, A Survey of Lie Groups and Lie Algebras with Applications and Computational Methods

James M. Ortega, Numerical Analysis: A Second Course

Anthony V. Fiacco and Garth P. McCormick, Nonlinear Programming: Sequential Unconstrained Minimization Techniques

F. H. Clarke, Optimization and Nonsmooth Analysis

George F. Carrier and Carl E. Pearson, Ordinary Differential Equations

Leo Breiman, Probability

R. Bellman and G. M. Wing, An Introduction to Invariant Imbedding

Abraham Berman and Robert J. Plemmons, Nonnegative Matrices in the Mathematical Sciences

Olvi L. Mangasarian, Nonlinear Programming

\*Carl Friedrich Gauss, Theory of the Combination of Observations Least Subject to Errors: Part One, Part Two, Supplement. Translated by G. W. Stewart

Richard Bellman, Introduction to Matrix Analysis

U. M. Ascher, R. M. M. Mattheij, and R. D. Russell, Numerical Solution of Boundary Value Problems for Ordinary Differential Equations

K. E. Brenan, S. L. Campbell, and L. R. Petzold, Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations

Charles L. Lawson and Richard J. Hanson, Solving Least Squares Problems

J. E. Dennis, Jr. and Robert B. Schnabel, Numerical Methods for Unconstrained Optimization and Nonlinear Equations

Richard E. Barlow and Frank Proschan, Mathematical Theory of Reliability

Cornelius Lanczos, Linear Differential Operators

Richard Bellman, Introduction to Matrix Analysis, Second Edition

<sup>\*</sup>First time in print.

#### Classics in Applied Mathematics (continued)

Beresford N. Parlett, The Symmetric Eigenvalue Problem

Richard Haberman, Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow

Peter W. M. John, Statistical Design and Analysis of Experiments

Tamer Başar and Geert Jan Olsder, Dynamic Noncooperative Game Theory, Second Edition

Emanuel Parzen, Stochastic Processes

Petar Kokotović, Hassan K. Khalil, and John O'Reilly, Singular Perturbation Methods in Control: Analysis and Design

Jean Dickinson Gibbons, Ingram Olkin, and Milton Sobel, Selecting and Ordering Populations: A New Statistical Methodology

James A. Murdock, Perturbations: Theory and Methods

Ivar Ekeland and Roger Témam, Convex Analysis and Variational Problems

Ivar Stakgold, Boundary Value Problems of Mathematical Physics, Volumes I and II

J. M. Ortega and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables

David Kinderlehrer and Guido Stampacchia, An Introduction to Variational Inequalities and Their Applications

F. Natterer, The Mathematics of Computerized Tomography

Avinash C. Kak and Malcolm Slaney, Principles of Computerized Tomographic Imaging

R. Wong, Asymptotic Approximations of Integrals

O. Axelsson and V. A. Barker, Finite Element Solution of Boundary Value Problems: Theory and Computation David R. Brillinger, Time Series: Data Analysis and Theory

Joel N. Franklin, Methods of Mathematical Economics: Linear and Nonlinear Programming, Fixed Point Theorems

Philip Hartman, Ordinary Differential Equations, Second Edition

Michael D. Intriligator, Mathematical Optimization and Economic Theory

Philippe G. Ciarlet, The Finite Element Method for Elliptic Problems

Jane K. Cullum and Ralph A. Willoughby, Lanczos Algorithms for Large Symmetric Eigenvalue Computations, Vol. I: Theory

M. Vidyasagar, Nonlinear Systems Analysis, Second Edition

Robert Mattheij and Jaap Molenaar, Ordinary Differential Equations in Theory and Practice

Shanti S. Gupta and S. Panchapakesan, Multiple Decision Procedures: Theory and Methodology of Selecting and Ranking Populations

Eugene L. Allgower and Kurt Georg, Introduction to Numerical Continuation Methods

Leah Edelstein-Keshet, Mathematical Models in Biology

Heinz-Otto Kreiss and Jens Lorenz, Initial-Boundary Value Problems and the Navier-Stokes Equations

J. L. Hodges, Jr. and E. L. Lehmann, Basic Concepts of Probability and Statistics, Second Edition

George F. Carrier, Max Krook, and Carl E. Pearson, Functions of a Complex Variable: Theory and Technique

Friedrich Pukelsheim, Optimal Design of Experiments

Israel Gohberg, Peter Lancaster, and Leiba Rodman, Invariant Subspaces of Matrices with Applications

Lee A. Segel with G. H. Handelman, Mathematics Applied to Continuum Mechanics

Rajendra Bhatia, Perturbation Bounds for Matrix Eigenvalues

Barry C. Arnold, N. Balakrishnan, and H. N. Nagaraja, A First Course in Order Statistics

Charles A. Desoer and M. Vidyasagar, Feedback Systems: Input-Output Properties

Stephen L. Campbell and Carl D. Meyer, Generalized Inverses of Linear Transformations

Copyright © 1991 by the Society for Industrial and Applied Mathematics.

This SIAM edition is an unabridged, corrected republication of the work first published by Blaisdell Publishing Company, Waltham, MA, 1968.

10987654

All rights reserved. Printed in the United States of America. No part of this book may be reproduced, stored, or transmitted in any manner without the written permission of the publisher. For information, write to the Society for Industrial and Applied Mathematics, 3600 Market Street, 6th Floor, Philadelphia, PA 19104-2688 USA.

#### Library of Congress Cataloging-in-Publication Data

Carrier, George F.

Ordinary differential equations / George F. Carrier, Carl E. Pearson.

p. cm. -- (Classics in applied mathematics: 6)

Originally published: Waltham, MA: Blaisdell Pub. Co., 1968.

Includes bibliographical references.

ISBN 978-0-898712-65-0

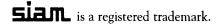
1. Differential equations. I. Pearson, Carl E. II. Title.

III. Series

QA372.C33 1991

515'.35--dc20

91-2642



#### **PREFACE**

The material in this book is not a conventional treatment of ordinary differential equations. It does not contain the collection of proofs commonly displayed as the foundations of the subject, nor does it contain the collection of recipes commonly aimed at the scientist or engineer. Rather, in a way which requires little or no previous acquaintance with the subject, it contains a sequence of heuristic arguments, illustrative examples and exercises, which serve to lead the reader towards the invention, generalization, and usage of the techniques by which solutions of differential equations can be constructed. Above all, we hope, the reader can gain a perspective concerning the extent to which methods which lead *in principle* to the solution of a given problem actually lead to a *useful* description of that solution.

Our purpose is to offer an alternative to the almost "rote" approach, in which the standard categories of differential equations, accompanied by routine problem sets, are systematically listed. We firmly believe that the present approach is one that should be encountered, at least once, by mathematicians, users of mathematics, and those who are merely curious about mathematics; we hope that members of all three sets will find the presentation stimulating.

We consider the exercises to be an essential part of the text. They extend, amplify, and provide perspective for the text material. (In rare cases, the statement of a problem is left deliberately incomplete, so as to give the reader some scope for ingenuity.) If this book is used to accompany a course of lectures, one technique would be to assign text reading and exercises beforehand, to be followed by classroom discussion and amplification. Alternatively, special problem sessions could be included.

In reprinting this book, we have taken the opportunity to correct an (embarrassingly large!) number of misprints, and to clarify certain aspects of the presentation. As in the original printing, the attitudes and approaches in this book remain solely the responsibility of the authors; however, we gratefully acknowledge the efforts of the many colleagues who have struggled to reform our viewpoints. In this connection, we are particularly appreciative of the suggestions provided by Bob Margulies and Frank Fendell, each of whom was kind enough to read portions of the present manuscript.

G.F.C. C.E.P.

### **CONTENTS**

INTRODUCTION 1	
Chapter 1 First-order linear differential equations	4
1.1 First-order linear ordinary differential equations 4 1.2 Problems 5 1.3 An illustrative sublimation problem 7 1.4 Problems 8 1.5 The nonhomogeneous equation 9 1.6 Problems 11 1.7 A nonlinear equation 13 1.8 Problems 13	
Chapter 2 First-order linear difference equations	16
2.1 Problems 18	
<b>Chapter 3</b> Second-order differential equations 20	
3.1 Problems 25 3.2 The homogeneous problem 28 3.3 Problems 29 3.4 Operators which can be factored 30 3.5 Problems 31	
Chapter 4 Power-series descriptions 33	
<ul> <li>4.1 Problems 35</li> <li>4.2 A more recalcitrant illustrative problem 36</li> <li>4.3 Problems 38</li> <li>4.4 Singular points 39</li> <li>4.5 Problems 40</li> <li>4.6 Singular points—continued 41</li> <li>4.7 Problems 42</li> </ul>	
Chapter 5 The Wronskian 43	
5.1 Problems 45	
Chapter 6 Eigenvalue Problems 47	
6.1 Problems 50 6.2 Fourier series 52 6.3 Problems 54 6.4 A special example 55 6.5 Problems 57	

<u>S</u>	x Contents
20110	Chapter 7 The second-order linear nonhomogeneous equation 5
m.org/terms-j	<ul> <li>7.1 Problems 61</li> <li>7.2 Green's functions 36</li> <li>7.3 Problems 68</li> <li>7.4 Nonhomogeneous boundary conditions 70</li> </ul>
5.SI	Chapter 8 Expansions in eigenfunctions 72
andə//:sdı	<ul><li>8.1 Problems 73</li><li>8.2 Two approximation methods 75</li><li>8.3 Problems 77</li></ul>
ee ee	Chapter 9 The perturbation expansion 79
opyrigni; s	<ul> <li>9.1 Problems 82</li> <li>9.2 An eigenvalue problem 82</li> <li>9.3 Problems 85</li> </ul>
5 5	Chapter 10 Asymptotic series 86
to 193.200.101.2 . Redistribution subject to MAM neemse of copyright, see futps://epuds.stain.org/teffins-privacy	10.1 Problems 88 10.2 An elementary technique 89 10.3 Problems 90 10.4 Another technique 91 10.5 Problems 93 10.6 Asymptotic expansions in a parameter 93 10.7 Problems 95  Chapter 11 Special functions 96
3.101.2 . Redistributio	11.1 The error function 96 11.2 The gamma function 98 11.3 Bessel functions of integral order 100 11.4 Bessel functions of nonintegral order 103 11.5 The Airy functions 104 11.6 The Legendre polynomials 105  Chapter 12 The Laplace transform 107
willoaded 09/04/23 to 193.200	12.1 Problems 107 12.2 Transform of a derivative 110 12.3 Problems 111 12.4 The convolution integral 114 12.5 Problems 115 12.6 Asymptotic behavior 116 12.7 Problems 117 12.8 Partial differential equations 118 12.9 Problems 119

Contents xi

										Contents
Chap	ter 13	Rudin	nents c	of the	variat	ional	calcul	us	121	
13.1	Problen	ns 12	4							
13.2	The sol	ution of	differe	ential e	equatio	ons	125			
13.3	Problem	ns 12			quan		1-0			
			-							
Chap	ter 14	Separa partial							s solu	tions of
14.1	The hea	at equati	on	130						
14.2		ns <sup>^</sup> 13								
14.3		nogeneo		ındarv	condi	tions	134			
14.4	Problem			J						
14.5		e's equa	_	136						
14.6	Problem	-		150						
14.7		r geome	-	139						
14.8	Problem		•	137						
14.9		lmholtz	-	Ωn	142					
	ter 15		•			anati	one	145		
Спар	ici 13	NOIIII	icai u	morei	mai C	quain	OHS	143		
15.1	Problen	ns 14	17							
15.2	Second	-order e	quation	ns; the	phase	plane	148	3		
15.3	Probler	ns 15	51		-	-				
15.5	Singula	r points	152	2						
15.6	_	ns 15								
Chap	ter 16	More	on dif	ferenc	e equ	ation	s 15	57		
16.1	Second	-order e	auatio	ns 1	57					
16.2			_							
16.3		neral lin	-	iation	160	)				
16.4	•		•	auton	100	,				
16.5		ting fun		163	}					
16.6	Problem	_		10.	,					
16.7		ear diffe	-	annatio	ne.	166				
16.7	Probler			equant	)115	100				
16.9				difforo	<b>n</b> 00 00	unation	ne 11	70		
		propert				uanoi	18 1	/0		
Chap	ter 17	Nume	rical n	netho	ds	174				
17.1	Probler	ns 17	78							
17.2		ization		ome o	ne-ste	p met	hods	179		
17.3	Probler		32							
17.4		Kutta n	_	183	}					
17.5	Probler		35	10.	-					
17.6		ep proc		netahi	lity	185				
17.0	Probler		38	nstavi.	ii iy	100				
17.7		oint bou		onditi	one: a	iaanu	alue pro	hlam	189	)
17.9	Probler		•	Jonaili	ons, c	igeliva	arue pre	JUICIII	102	r
			-							

Index

219

xii Contents	
Chapter 18 Singular perturbation methods	193
18.1 The boundary layer idea 193	
18.2 Problems 199	
18.3 Interior layers and a spurious construction	202
18.4 Problems 205	
18.5 The turning-point problem 205	
18.6 Problems 209	
18.7 Two-timing 210	
18.8 Problems 216	