

Applied Analysis by the Hilbert Space Method

**An Introduction with Applications to the
Wave, Heat, and Schrödinger Equations**

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Contents

<i>Preface</i>	v
<i>A Note on Method</i>	vii
Chapter 1. First Order Linear Differential Equations	1
1.1. The Equation $a(x)y' + b(x)y = h(x)$	1
1.2. First Order Linear Differential Expressions; the Kernel	4
1.3. Finding a Particular Solution by Variation of Parameters	9
1.4. Power Series Review	10
1.5. The Initial Value Problem for a First Order Linear Differential Equation	18
Chapter 2. Second Order Linear Differential Equations	22
2.1. Basic Concepts of Linear Algebra for Function Spaces	22
2.2. The Initial Value Problem for a Second Order Linear Homogeneous Differential Equation	28
2.3. Dimension of the Kernel; General Solution; Abel's Formula	35
2.4. Kernel of Constant-Coefficient Expressions	39
2.5. The Classical Linear Oscillator	43
2.6. Guessing a Particular Solution to a Constant-Coefficient Equation	48
2.7. Particular Solution by Variation of Parameters	51
2.8. The Kernel of Legendre's Differential Expression	53
	ix

2.9.	The Kernels of Other Classical Expressions	57
2.10.	Dirac's Delta Function and Green's Functions	59
	Appendix 2.A. Second Order Linear Differential Equations in the Complex Domain	69
Chapter 3.	Hilbert Space	78
3.1.	The Vibrating Wire	78
3.2.	Fourier Series	83
3.3.	Fourier Sine and Cosine Series	94
3.4.	Fourier Series over Other Intervals	99
3.5.	The Vibrating Wire, Revisited	102
3.6.	The Inner Product	108
3.7.	Schwarz's Inequality	118
3.8.	The Mean-Square Metric; Orthogonal Bases	127
3.9.	L^2 Spaces	138
3.10.	Hilbert Space	149
Chapter 4.	Linear Second Order Differential Operators in L^2 Spaces and Their Eigenvalues and Eigenfunctions	157
4.1.	Compatibility	157
4.2.	Eigenvalues and Eigenfunctions	168
4.3.	Hermitian Operators	175
4.4.	Some General Operator Theory	184
4.5.	The One-Dimensional Laplacian	191
4.6.	Legendre's Operator and Its Eigenfunctions, the Legendre Polynomials	202
4.7.	Solving Operator Equations with Legendre's Operator	217
4.8.	More on Legendre Polynomials: Rodrigues' Formula, the Recursion Relation, and the Generating Function	224
4.9.	Hermite's Operator and Its Eigenfunctions, the Hermite Polynomials	233
4.10.	Solving Operator Equations with Hermite's Operator	243
4.11.	More on Hermite Polynomials: Rodrigues' Formula, the Recursion Relation, and the Generating Function	248
	Appendix 4.A. Mathematical Aspects of Differential Operators in L^2 Spaces	253
Chapter 5.	Schrödinger's Equations in One Dimension	267
5.1.	The Wave Equation by the Hilbert Space Method	267
5.2.	The Heat Equation by the Hilbert Space Method	273
5.3.	Quanta as Eigenvalues—the Time-Independent Schrödinger Equation in One Dimension	282

5.4.	Interpretation of the Ψ Function. The Time-Dependent Schrödinger Equation in One Dimension	289
5.5.	The Quantum Linear Oscillator	298
5.6.	Solution of the Time-Dependent Schrödinger Equation	308
5.7.	A Brief History of Matrix Mechanics	313
5.8.	A General Formulation of Quantum Mechanics: States	316
5.9.	A General Formulation of Quantum Mechanics: Observables	326
Chapter 6.	Bessel's Operator and Bessel Functions	331
6.1.	The Wave Equation and Other Equations in Higher Dimensions; Polar Coordinates	331
6.2.	Bessel's Equation and Bessel's Operator of Order Zero	339
6.3.	$J_0(x)$: The Bessel Function of the First Kind of Order Zero	345
6.4.	$J_0(x)$: Calculating Its Values and Finding Its Zeros	350
6.5.	The Eigenvalues and Eigenfunctions of Bessel's Operator of Order Zero	359
6.6.	The Vibrating Drumhead, the Heated Disk, and the Quantum Particle Confined to a Circular Region (the θ -Independent Case)	367
6.7.	θ Dependence: Bessel's Equation and Bessel's Operator of Integral Order p	375
6.8.	$J_p(x)$: The Bessel Functions of the First Kind of Integral Order p	380
6.9.	The Eigenvalues and Eigenfunctions of Bessel's Operator of Integral Order p	387
6.10.	Project on Bessel Functions of Nonintegral Order	392
	Appendix 6A. Mathematical Theory of Bessel's Operator	393
Chapter 7.	Eigenvalues of the Laplacian, with Applications	402
7.1.	The Laplacian as a Hilbert Space Operator	402
7.2.	Differential Forms, the Stokes Theorem, and Integration by Parts in Two Variables	410
7.3.	The Laplacian Is Hermitian	418
7.4.	General Facts About the Eigenvalues of the Laplacian	426
7.5.	Eigenvalues of the Rectangle	440
7.6.	Eigenvalues of the Disk	450
7.7.	The Laplacian on the Sphere	452
7.8.	Eigenvalues of the Sphere; Spherical Harmonics	466
7.9.	The Hydrogen Atom	481
7.10.	Project on Laguerre's Operator and Laguerre Polynomials	493

7.11. Laplace's Equation and Harmonic Polynomials	500
Appendix 7.A. The Legendre, Laguerre, and Schrödinger Operators	507
Chapter 8. The Fourier Transform	514
8.1. Complex Methods in Fourier Series; the Fourier Transform	514
8.2. Plancherel's Theorem; Examples of Fourier Transforms	520
8.3. Fourier Sine and Cosine Transforms	526
8.4. The Fourier Transform Is a Unitary Operator on $L^2(-\infty, \infty)$.	529
8.5. The Fourier Transform Converts Differentiation into Multiplication by the Independent Variable	537
8.6. The Eigenvalues and Eigenfunctions of the Fourier Transform	544
Appendix 8.A. $\int_{-\infty}^{\infty} ((\sin x)/x) dx = \pi$	551
<i>Index of Symbols</i>	553
<i>Index</i>	555