

**Mathematical
Surveys
and
Monographs**
Volume 232

Linear Holomorphic Partial Differential Equations and Classical Potential Theory

**Dmitry Khavinson
Erik Lundberg**



Contents

Preface	ix
Acknowledgments	x
Chapter 1. Introduction: Some Motivating Questions	1
1. Continuation of potentials	1
2. Uniqueness of potentials	2
3. The Schwarz reflection principle	3
4. Szegő's theorem	3
5. PDE vs. ODE	4
6. Laplacian growth and the inverse potential problem	5
7. Some basic notation	5
Notes	6
Chapter 2. The Cauchy-Kovalevskaya Theorem with Estimates	7
1. Proof of uniqueness	7
2. Proof of existence	8
3. Proofs of accessory lemmas: Fun and useful inequalities	10
Notes	12
Chapter 3. Remarks on the Cauchy-Kovalevskaya Theorem	13
1. The Cauchy problem with holomorphic data	13
2. Transversality of the highest-order derivatives	14
3. The C-K theorem for non-singular hypersurfaces	15
4. The Goursat problem	17
5. Existence of the Riemann function	18
Notes	18
Chapter 4. Zerner's Theorem	19
1. Real and complex hyperplanes	19
2. Zerner characteristic hypersurfaces	20
3. Proof of Zerner's theorem	21
4. A corollary: The Delassus-Le Roux theorem	22
Notes	23
Chapter 5. The Method of Globalizing Families	25
1. Globalizing families	25
2. The globalizing principle	25
3. Applications	25
Notes	27
Chapter 6. Holmgren's Uniqueness Theorem	29

1. A uniqueness result for harmonic functions	29
2. Holmgren's uniqueness theorem	30
Notes	33
 Chapter 7. The Continuity Method of F. John	 35
1. A global uniqueness result	35
2. Exercises	36
Notes	37
 Chapter 8. The Bony-Schapira Theorem	 39
1. Applications of the Bony-Schapira theorem	39
2. Proof of the Bony-Schapira theorem	41
3. Exercises	42
Notes	43
 Chapter 9. Applications of the Bony-Schapira Theorem:	
Part I - Vekua Hulls	45
1. A uniqueness question for harmonic functions	45
2. A view from C^1 : The Vekua hull	48
3. Is the connectivity condition also necessary?	54
Notes	56
 Chapter 10. Applications of the Bony-Schapira Theorem:	
Part II - Szegő's Theorem Revisited	57
1. Jacobi polynomial expansions: Generalization of Szegő's theorem	58
2. Relation to holomorphic PDEs	60
3. Proof of the generalized Szegő theorem	61
4. Nehari's theorem revisited	64
Notes	70
 Chapter 11. The Reflection Principle	 73
1. The Schwarz function of a curve	73
2. E. Study's interpretation of the Schwarz reflection principle	75
3. Failure of the reflection law for other operators	76
Notes	81
 Chapter 12. The Reflection Principle (continued)	 83
1. The Study relation	83
2. Reflection in higher dimensions	86
3. The even-dimensional case	90
4. The odd-dimensional case	95
Notes	97
 Chapter 13. Cauchy Problems and the Schwarz Potential Conjecture	 99
1. Analytic continuation of potentials and quadrature domains	101
2. The Schwarz potential conjecture	103
Notes	106
 Chapter 14. The Schwarz Potential Conjecture for Spheres	 107
Notes	114

Chapter 15. Potential Theory on Ellipsoids: Part I - The Mean Value Property	115
1. Proof of MacLaurin's theorem using E. Fischer's inner product	116
2. The Newtonian potential of an ellipsoid	119
Notes	122
Chapter 16. Potential Theory on Ellipsoids: Part II - There is No Gravity in the Cavity	123
1. Arbitrary polynomial density	123
2. The standard single layer potential	125
3. Domains of hyperbolicity	127
4. The Schwarz potential conjecture for ellipsoids	128
Notes	130
Chapter 17. Potential Theory on Ellipsoids: Part III - The Dirichlet Problem	133
1. The Dirichlet problem in an ellipsoid: Polynomial data	133
2. Entire data	134
3. The Khavinson-Shapiro conjecture	136
4. The Brelot-Choquet theorem and harmonic divisors	137
Notes	137
Chapter 18. Singularities Encountered by the Analytic Continuation of Solutions to the Dirichlet Problem	139
1. The Dirichlet problem: When does entire data imply entire solution?	140
2. When does polynomial data imply polynomial solution?	140
3. The Dirichlet problem and Bergman orthogonal polynomials	142
4. Singularities of the solutions to the Dirichlet problem	142
5. Render's theorem	144
6. Back to \mathbb{R}^2 : Annihilating measures and closed lightning bolts	146
Notes	149
Chapter 19. An Introduction to J. Leray's Principle on Propagation of Singularities through \mathbb{C}^n	151
1. Introductory remarks on propagation of singularities	151
2. Local propagation of singularities in \mathbb{C}^n : Leray's principle	154
Notes	165
Chapter 20. Global Propagation of Singularities in \mathbb{C}^n	167
1. Global propagation of singularities and Poincaré equations	167
2. A note on characteristic surfaces for the Laplace operator	176
Notes	178
Chapter 21. Quadrature Domains and Laplacian Growth	181
1. Dynamics of singularities of the Schwarz potential	182
2. Quadrature domains and Richardson's theorem	183
3. Exact solutions in the plane	185
4. Algebraicity of planar quadrature domains	186
5. Higher-dimensional quadrature domains need not be algebraic	186
Notes	193

Chapter 22. Other Varieties of Quadrature Domains	195
1. Ellipsoids as quadrature domains in the wide sense	195
2. Null quadrature domains	196
3. Arclength quadrature domains	196
4. Lemniscates as quadrature domains for equilibrium measure	197
5. Quadrature domains for other classes of test functions	199
Notes	201
Bibliography	203
Index	213