

# **DISCRETE CHAOS**

## **Second Edition**

---

**WITH APPLICATIONS IN  
SCIENCE AND ENGINEERING**

**Saber N. Elaydi**

Trinity University  
San Antonio, Texas, U.S.A.



**Chapman & Hall/CRC**  
Taylor & Francis Group

Boca Raton London New York

---

Chapman & Hall/CRC is an imprint of the  
Taylor & Francis Group, an Informa business

---

## *Contents*

Preface	ix
Author Biography	xv
List of Symbols	xvii
Foreword	xix
<b>1 The Stability of One-Dimensional Maps</b>	<b>1</b>
1.1 Introduction . . . . .	2
1.2 Maps vs. Difference Equations . . . . .	2
1.3 Maps vs. Differential Equations . . . . .	4
1.3.1 Euler's Method . . . . .	4
1.3.2 Poincaré Map . . . . .	8
1.4 Linear Maps/Difference Equations . . . . .	9
1.5 Fixed (Equilibrium) Points . . . . .	14
1.6 Graphical Iteration and Stability . . . . .	19
1.7 Criteria for Stability . . . . .	25
1.7.1 Hyperbolic Fixed Points . . . . .	25
1.7.2 Nonhyperbolic Fixed Points . . . . .	28
1.8 Periodic Points and Their Stability . . . . .	36
1.9 The Period-Doubling Route to Chaos . . . . .	43
1.9.1 Fixed Points . . . . .	43
1.9.2 2-Periodic Cycles . . . . .	44
1.9.3 2 <sup>2</sup> -Periodic Cycles . . . . .	46
1.9.4 Beyond $\mu_{\infty}$ . . . . .	50
1.10 Applications . . . . .	54
1.10.1 Fish Population Modeling . . . . .	54
<b>2 Attraction and Bifurcation</b>	<b>61</b>
2.1 Introduction . . . . .	61
2.2 Basin of Attraction of Fixed Points . . . . .	62
2.3 Basin of Attraction of Periodic Orbits . . . . .	66
2.4 Singer's Theorem . . . . .	70
2.5 Bifurcation . . . . .	81
2.6 Sharkovsky's Theorem . . . . .	94
2.6.1 Li-Yorke Theorem . . . . .	94

2.6.2	A Converse of Sharkovsky's Theorem . . . . .	99
2.7	The Lorenz Map . . . . .	105
2.8	Period-Doubling in the Real World . . . . .	108
2.9	Poincaré Section/Map . . . . .	110
2.9.1	. . . . .	110
2.9.2	Belousov-Zhabotinskii Chemical Reaction . . . . .	111
<b>Appendix</b>	. . . . .	115
<b>3 Chaos in One Dimension</b>	. . . . .	<b>119</b>
3.1	Introduction . . . . .	119
3.2	Density of the Set of Periodic Points . . . . .	120
3.3	Transitivity . . . . .	124
3.4	Sensitive Dependence . . . . .	129
3.5	Definition of Chaos . . . . .	137
3.6	Cantor Sets . . . . .	144
3.7	Symbolic Dynamics . . . . .	149
3.8	Conjugacy . . . . .	154
3.9	Other Notions of Chaos . . . . .	161
3.10	Rössler's Attractor . . . . .	163
3.11	Saturn's Rings . . . . .	167
<b>4 Stability of Two-Dimensional Maps</b>	. . . . .	<b>171</b>
4.1	Linear Maps vs. Linear Systems . . . . .	171
4.2	Computing $A^n$ . . . . .	172
4.3	Fundamental Set of Solutions . . . . .	179
4.4	Second-Order Difference Equations . . . . .	181
4.5	Phase Space Diagrams . . . . .	184
4.6	Stability Notions . . . . .	192
4.7	Stability of Linear Systems . . . . .	197
4.8	The Trace-Determinant Plane . . . . .	200
4.8.1	Stability Analysis . . . . .	200
4.8.2	Navigating the Trace-Determinant Plane . . . . .	204
4.9	Liapunov Functions for Nonlinear Maps . . . . .	207
4.10	Linear Systems Revisited . . . . .	215
4.11	Stability via Linearization . . . . .	219
4.11.1	The Hartman-Grobman Theorem . . . . .	224
4.11.2	The Stable Manifold Theorem . . . . .	225
4.12	Applications . . . . .	228
4.12.1	The Kicked Rotator and the Hénon Map . . . . .	228
4.12.2	The Hénon Map . . . . .	230
4.12.3	Discrete Epidemic Model for Gonorrhea . . . . .	233
4.12.4	Perennial Grass . . . . .	236
<b>Appendix</b>	. . . . .	239

<b>5 Bifurcation and Chaos in Two Dimensions</b>	<b>241</b>
5.1 Center Manifolds . . . . .	241
5.2 Bifurcation . . . . .	248
5.2.1 Eigenvalues of 1 or -1 . . . . .	248
5.2.2 A Pair of Eigenvalues of Modulus 1 - The Neimark-Sacker Bifurcation . . . . .	249
5.3 Hyperbolic Anosov Toral Automorphism . . . . .	257
5.4 Symbolic Dynamics . . . . .	262
5.4.1 Subshifts of Finite Type . . . . .	263
5.5 The Horseshoe and Hénon Maps . . . . .	268
5.5.1 The Hénon Map . . . . .	272
5.6 A Case Study: The Extinction and Sustainability in Ancient Civilizations . . . . .	278
<b>Appendix . . . . .</b>	<b>285</b>
<b>6 Fractals</b>	<b>289</b>
6.1 Examples of Fractals . . . . .	289
6.2 L-system . . . . .	299
6.3 The Dimension of a Fractal . . . . .	300
6.4 Iterated Function System . . . . .	314
6.4.1 Deterministic IFS . . . . .	315
6.4.2 The Random Iterated Function System and the Chaos Game, . . . . .	325
6.5 Mathematical Foundation of Fractals . . . . .	330
6.6 The Collage Theorem and Image Compression . . . . .	338
<b>7 The Julia and Mandelbrot Sets</b>	<b>341</b>
7.1 Introduction . . . . .	341
7.2 Mapping by Functions on the Complex Domain . . . . .	342
7.3 The Riemann Sphere . . . . .	351
7.4 The Julia Set . . . . .	354
7.5 Topological Properties of the Julia Set . . . . .	364
7.6 Newton's Method in the Complex Plane . . . . .	371
7.7 The Mandelbrot Set . . . . .	375
7.7.1 Topological Properties . . . . .	375
7.7.2 Rays and Bulbs . . . . .	377
7.7.3 Rotation Numbers and Farey Addition . . . . .	380
7.7.4 Accuracy of Pictures . . . . .	385
<b>Bibliography</b>	<b>389</b>
<b>Answers to Selected Problems</b>	<b>397</b>
<b>Index</b>	<b>414</b>