

Norman R. Howes

Modern Analysis and Topology



Springer-Verlag

New York Berlin Heidelberg London Paris
Tokyo Hong Kong Barcelona Budapest

Norman R. Howes
Institute for Defense Analyses
1801 N. Beauregard Street
Alexandria, VA 22311-1772
USA

Editorial Board
(North America):

J.H. Ewing
Department of Mathematics
Indiana University
Bloomington, IN 47405
USA

F.W. Gehring
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
USA

P.R. Halmos
Department of Mathematics
Santa Clara University
Santa Clara, CA 95053
USA

Mathematics Subject Classifications (1991): 26-02, 54-02, 54D20, 54D60, 54E15, 28A05, 28C10, 46Exx

Library of Congress Cataloging-in-Publication Data
Howes, Norman R.

Modern analysis and topology / Norman R. Howes.

p. cm. — (Universitext)

Includes bibliographical references (p. —) and index.

ISBN 0-387-97986-7 (softcover : acid-free)

1. Mathematical analysis. 2. Topology I. Title.

QA300.H69 1995

515'.13—dc20

95-3995

Printed on acid-free paper.

© 1995 Springer-Verlag New York, Inc.

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by Bill Imbornoni; manufacturing supervised by Joe Quatela.

Camera-ready copy prepared by the author.

Printed and bound by R.R. Donnelley & Sons, Harrisonburg, VA.

Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-97986-7 Springer-Verlag New York Berlin Heidelberg

CONTENTS

PREFACE	vii
INTRODUCTION: TOPOLOGICAL BACKGROUND	xvii
CHAPTER 1: METRIC SPACES	1
1.1 Metric and Pseudo-Metric Spaces	1
Distance Functions, Spheres, Topology of Pseudo-Metric Spaces, The Ring $C^*(X)$, Real Hilbert Space, The Distance from a Point to a Set, Partitions of Unity	
1.2 Stone's Theorem	6
Refinements, Star Refinements and Δ -Refinements, Full Normality, Paracompactness, Shrinkable Coverings, Stone's Theorem	
1.3 The Metrization Problem	13
Functions That Can Distinguish Points from Sets, σ -Local Finiteness, Urysohn's Metrization Theorem, The Nagata-Smirnov Metrization Theorem, Local Starrings, Arhangel'skii's Metrization Theorem	
1.4 Topology of Metric Spaces	20
Complete Normality and Perfect Normality, First and Second Countable Spaces, Separable Spaces, The Diameter of a Set, The Lebesgue Number, Precompact Spaces, Countably Compact and Sequentially Compact Spaces	
1.5 Uniform Continuity and Uniform Convergence	25
Uniform Continuity, Uniform Homeomorphisms and Isomorphisms, Isometric Functions, Uniform Convergence	
1.6 Completeness	28
Convergence and Clustering of Sequences, Cauchy Sequences and Cofinally Cauchy, Sequences, Complete and Cofinally Complete Spaces, The Lebesgue Property, Borel Compactness, Regularly Bounded Metric Spaces	
1.7 Completions	38
The Completion of a Metric Space, Uniformly Continuous Extensions	
CHAPTER 2: UNIFORMITIES	43
2.1 Covering Uniformities	43
Uniform Spaces, Normal Sequences of Coverings, Bases and Subbases for Uniformities, Normal Coverings, Uniform Topology	

2.2	Uniform Continuity	48
	Uniform Continuity, Uniform Homeomorphisms, Pseudo-Metrics Determined by Normal Sequences	
2.3	Uniformizability and Complete Regularity	52
	Uniformizable Spaces, The Equivalence of Uniformizability and Complete Regularity, Regularly Open Sets and Coverings, Open and Closed Bases of Uniformities, Regularly Open Bases of Uniformities, Universal or Fine Uniformities	
2.4	Normal Coverings	56
	The Unique Uniformity of a Compact Hausdorff Space, Tukey's Characterization of Normal Spaces, Star-Finite Coverings, Precise Refinements, Some Results of K. Morita, Some Corrections of Tukey's Theorems by Morita	
CHAPTER 3: TRANSFINITE SEQUENCES		62
3.1	Background	62
3.2	Transfinite Sequences in Uniform Spaces	63
	Cauchy and Cofinally Cauchy Transfinite Sequences, A Characterization of Paracompactness in Terms of Transfinite Sequences, Shirota's ϵ Uniformity, Some Characterizations of the Lindelöf Property in Terms of Transfinite Sequences, The β Uniformity, A Characterization of Compactness in Terms of Transfinite Sequences	
3.3	Transfinite Sequences and Topologies	75
	Characterizations of Open and Closed Sets in Terms of Transfinite Sequences, A Characterization of the Hausdorff Property in Terms of Transfinite Sequences, Cluster Classes and the Characterization of Topologies, A Characterization of Continuity in Terms of Transfinite Sequences	
CHAPTER 4: COMPLETENESS, COFINAL COMPLETENESS AND UNIFORM PARACOMPACTNESS		83
4.1	Introduction	83
4.2	Nets	84
	Convergence and Clustering of Nets, Characterizations of Open and Closed Sets in Terms of Nets, A Characterization of the Hausdorff Property in Terms of Nets, Subnets, A Characterization of	

Compactness in Terms of Nets, A Characterization of Continuity in Terms of Nets, Convergence Classes and the Characterization of Topologies, Universal Nets, Characterizations of Paracompactness, the Lindelöf Property and Compactness in Terms of Nets	
4.3 Completeness, Cofinal Completeness and Uniform Paracompactness	92
Cauchy and Cofinally Cauchy Nets, Completeness and Cofinal Completeness, The Lebesgue Property, Precompactness, Uniform Paracompactness	
4.4 The Completion of a Uniform Space	97
Fundamental Nets, Completeness in Terms of Fundamental Nets, The Construction of the Completion with Fundamental Nets, The Uniqueness of the Completion	
4.5 The Cofinal Completion or Uniform Paracompactification	103
The Topological Completion, Preparacompactness, Countable Boundedness and the Lindelöf Property, A Necessary and Sufficient Condition for a Uniform Space to Have a Paracompact Completion, A Necessary and Sufficient Condition for a Uniform Space to Have a Lindelöf Completion, The Existence of the Cofinal Completion, A Characterization of Preparacompactness	
Chapter 5: FUNDAMENTAL CONSTRUCTIONS	110
5.1 Introduction	110
5.2 Limit Uniformities	111
Infimum and Supremum Topologies, Infimum and Supremum Uniformities, Projective and Inductive Limit Topologies, Projective and Inductive Limit Uniformities	
5.3 Subspaces, Sums, Products and Quotients	114
Uniform Product Spaces, Uniform Subspaces, Quotient Uniform Spaces, The Uniform Sum	
5.4 Hyperspaces	119
The Hyperspace of a Uniform Space, Supercompleteness, Burdick's Characterization of Supercompleteness, Other Characterizations of Supercompleteness, Supercompleteness and Cofinal Completeness, Paracompactness and Supercompleteness	

5.5 Inverse Limits and Spectra	126
Inverse Limit Sequences, Inverse Limit Systems, Inverse Limit Systems of Uniform Spaces, Morita's Weak Completion, The Spectrum of Weakly Complete Uniform Spaces, Morita's and Pasynkov's Characterizations of Closed Subsets of Products of Metric Spaces	
5.6 The Locally Fine Coreflection	133
Uniformly Locally Uniform Coverings, Locally Fine Uniform Spaces, The Derivative of a Uniformity, Partially Cauchy Nets, Injective Uniform Spaces, Subfine Uniform Spaces, The Subfine Coreflection	
5.7 Categories and Functors	146
Concrete Categories, Objects, Morphisms, Covariant Functors, Isomorphisms, Monomorphisms, Duality, Subcategories, Reflection, Coreflection	
CHAPTER 6: PARACOMPACTIFICATIONS	156
6.1 Introduction	156
Some Problems of K. Morita and H. Tamano, Topological Completion, Paracompactifications, Compactifications, Samuel Compactifications, The Stone-Čech Compactification, Uniform Paracompactifications, Tamano's Paracompactification Problem	
6.2 Compactifications	159
Extensions of Open Sets, Extensions of Coverings, The Extent of a Covering, Stable Coverings, Star-Finite Partitions of Unity	
6.3 Tamano's Completeness Theorem	171
The Radical of a Uniform Space, Tamano's Completeness Theorem, Necessary and Sufficient Conditions for Topological Completeness	
6.4 Points at Infinity and Tamano's Theorem	178
Points and Sets at Infinity, Some Characterizations of Paracompactness by Tamano, Tamano's Theorem	
6.5 Paracompactifications	182
Completions of Uniform Spaces as Subsets of βX , A Solution of Tamano's Paracompactification Problem, The Tamano-Morita Paracompactification, Characterizations of Paracompactness, the Lindelöf Property and Compactness in Terms of Supercompleteness, Another Necessary and Sufficient Condition for a Uniform Space to Have a Paracompact Completion, Another Necessary and Sufficient Condition	

for a Uniform Space to Have a Lindelöf Completion, The Definition and Existence of the Supercompletion	
6.6 The Spectrum of βX	192
The Spectrum of βX , The Spectrum of uX , Morita's Weak Completion	
6.7 The Tamano-Morita Paracompactification	197
M-spaces, Perfect and Quasi-perfect Mappings, The Topological Completion of an M-space, The Tamano-Morita Paracompactification of an M-space	
 CHAPTER 7: REALCOMPACTIFICATIONS	 202
7.1 Introduction	202
Another Characterization of βX , Q-spaces, CZ-maximal Families	
7.2 Realcompact Spaces	203
Realcompact Spaces, The Hewitt Realcompactification, Characterizations of Realcompactness, Properties of Realcompact Spaces, Pseudo-metric Uniformities, The c and c^* Uniformities	
7.3 Realcompactifications	210
Realcompactifications, The Equivalence of νX and eX , The Uniqueness of the Hewitt Realcompactification, Characterizations of νX , Properties of νX , Hereditary Realcompactness	
7.4 Realcompact Spaces and Lindelöf Spaces	217
Tamano's Characterization of Realcompact Spaces, A Necessary and Sufficient Condition for the Realcompactification to be Lindelöf, Tamano's Characterization of Lindelöf Spaces	
7.5 Shirota's Theorem	221
Measurable Cardinals, $\{0,1\}$ Measures, The Relationship of Non-Zero $\{0,1\}$ Measures and CZ-maximal Families, A Necessary and Sufficient Condition for Discrete Spaces to be Realcompact, Closed Classes of Cardinals, Shirota's Theorem	
 CHAPTER 8: MEASURE AND INTEGRATION	 229
8.1 Introduction	229
Riemann Integration, Lebesgue Integration, Measures, Invariant Integrals	

8.2	Measure Rings and Algebras	230
	Rings, Algebras, σ -Rings, σ -Algebras, Borel Sets, Baire Sets, Measures, Measure Rings, Measurable Sets, Measure Algebras, Measure Spaces, Complete Measures, The Completion of a Measure, Borel Measures, Lebesgue Measure, Baire Measures, The Lebesgue Ring, Lebesgue Measurable Sets, Finite Measures, Infinite Measures	
8.3	Properties of Measures	235
	Monotone Collections, Continuous from Below, Continuous from Above	
8.4	Outer Measures	238
	Hereditary Collections, Outer Measures, Extensions of Measures, μ^* -Measurability	
8.5	Measurable Functions	243
	Measurable Spaces, Measurable Sets, Measurable Functions, Borel Functions, Limits Superior, Limits Inferior, Point-wise Limits of Functions, Simple Functions, Simple Measurable Functions	
8.6	The Lebesgue Integral	249
	Development of the Lebesgue Integral	
8.7	Negligible Sets	256
	Negligible Sets, Almost Everywhere, Complete Measures, Completion of a Measure	
8.8	Linear Functionals and Integrals	257
	Linear Functionals, Positive Linear Functionals, Lower Semi-continuous, Upper Semi-continuous, Outer Regularity, Inner Regularity, Regular Measures, Almost Regular Measures, The Riesz Representation Theorem	
CHAPTER 9: HAAR MEASURE IN UNIFORM SPACES		264
9.1	Introduction	264
	Isogeneous Uniform Spaces, Isomorphisms, Homogeneous Spaces, Translations, Rotations, Reflections, Haar Integral, Haar Measure	
9.2	Haar Integrals and Measures	267
	Development of the Haar integral on Locally Compact Isogeneous Uniform Spaces	

9.3 Topological Groups and Uniqueness of Haar Measures	271
<p>Topological Groups, Abelian Topological Groups, Open at 0, Right Uniformity, Left Uniformity, Right Coset, Left Coset, Quotient of a Topological Group, A Necessary and Sufficient Condition for a Locally Compact Space to Have a Topological Group Structure</p>	
CHAPTER 10: UNIFORM MEASURES	284
10.1 Introduction	284
<p>Uniform Measures, The Congruence Axiom, Loomis Contents</p>	
10.2 Prerings and Loomis Contents	285
<p>Prerings, Hereditary Open Prerings, Loomis Contents, Uniformly Separated, Left Continuity, Invariant Loomis Contents, Zero-boundary Sets</p>	
10.3 The Haar Functions	292
<p>The Haar Covering Function, The Haar Function, Extension of Loomis Contents to Finitely Additive Measures</p>	
10.4 Invariance and Uniqueness of Loomis Contents and Haar Measures	299
<p>Invariance with Respect to a Uniform Covering, Invariance on Compact Spheres, Development of Loomis Contents on Suitably Restricted Uniform Spaces.</p>	
10.5 Local Compactness and Uniform Measures	304
<p>Almost Uniform Measures, Uniform Measures, Jordan Contents, Monotone Sequences of Sets, Monotone Classes, Development of Uniform Measures on Suitably Restricted Uniform Spaces</p>	
CHAPTER 11: SPACES OF FUNCTIONS	317
11.1 L^p -spaces	317
<p>Conjugate Exponents, L^p-norm, The Essential Supremum, Essentially Bounded, Minkowski's Inequality, Hölder's Inequality, The Supremum Norm, The Completion of $C_K(X)$ with Respect to the L^p-norm</p>	
11.2 The Space $L^2(\mu)$ and Hilbert Spaces	326
<p>Square Integrable Functions, Inner Product, Schwarz Inequality, Hilbert Space, Orthogonality, Orthogonal Projections, Linear Combinations, Linear Independence, Span, Basis of a Vector Space,</p>	

Orthonormal Sets, Orthonormal Bases, Bessel's Inequality, Riesz-Fischer Theorem, Hilbert Space Isomorphism	
11.3 The Space $L^p(\mu)$ and Banach Spaces	340
Normed Linear Space, Banach Space, Linear Operators, Kernel of a Linear Operator, Bounded Linear Operators, Dual Spaces, Hahn-Banach Theorem, Second Dual Space, Baire's Category Theorem, Nowhere Dense Sets, Open Mapping Theorem, Closed Graph Theorem, Uniform Boundedness Principle, Banach-Steinhaus Theorem	
11.4 Uniform Function Spaces	355
Uniformity of Pointwise Convergence, Uniformity of Uniform Convergence, Joint Continuity, Uniformity of Uniform Convergence on Compacta, Topology of Compact Convergence, Compact-Open Topology, Joint Continuity on Compacta, Ascoli Theorem, Equicontinuity	
CHAPTER 12: UNIFORM DIFFERENTIATION	370
12.1 Complex Measures	370
Complex Measure, Total Variation, Absolute Continuity, Concentration of a Measure on a Subset, Orthogonality of Measures	
12.2 The Radon-Nikodym Derivative	373
Radon-Nikodym Derivative and its Applications	
12.3 Decompositions of Measures and Complex Integration	380
Polar Decomposition, Lebesgue Decomposition, Complex Integration	
12.4 The Riesz Representation Theorem	386
Regular and Almost Regular Complex Measures, The Riesz Representation Theorem	
12.5 Uniform Derivatives of Measures	389
Differentiation of a Measure at a Point, Differentiable Measures, L^1 -differentiable Measures, Uniformly Differentiable Measures, Fubini's Theorem	
INDEX	394