Texts in Applied Mathematics 20



**Springer**
*New York*
*Berlin*
*Heidelberg*
*Barcelona*
*Budapest*
*Hong Kong*
*London*
*Milan*
*Paris*
*Singapore*
*Tokyo*

# Texts in Applied Mathematics

1. *Sirovich:* Introduction to Applied Mathematics.
2. *Wiggins:* Introduction to Applied Nonlinear Dynamical Systems and Chaos.
3. *Hale/Koçak:* Dynamics and Bifurcations.
4. *Chorin/Marsden:* A Mathematical Introduction to Fluid Mechanics, 3rd ed.
5. *Hubbard/West:* Differential Equations: A Dynamical Systems Approach: Ordinary Differential Equations.
6. *Sontag:* Mathematical Control Theory: Deterministic Finite Dimensional Systems, 2nd ed.
7. *Perko:* Differential Equations and Dynamical Systems, 2nd ed.
8. *Seaborn:* Hypergeometric Functions and Their Applications.
9. *Pipkin:* A Course on Integral Equations.
10. *Hoppensteadt/Peskin:* Mathematics in Medicine and the Life Sciences.
11. *Braun:* Differential Equations and Their Applications, 4th ed.
12. *Stoer/Bulirsch:* Introduction to Numerical Analysis, 2nd ed.
13. *Renardy/Rogers:* A First Graduate Course in Partial Differential Equations.
14. *Banks:* Growth and Diffusion Phenomena: Mathematical Frameworks and Applications.
15. *Brenner/Scott:* The Mathematical Theory of Finite Element Methods.
16. *Van de Velde:* Concurrent Scientific Computing.
17. *Marsden/Ratiu:* Introduction to Mechanics and Symmetry.
18. *Hubbard/West:* Differential Equations: A Dynamical Systems Approach: Higher-Dimensional Systems.
19. *Kaplan/Glass*: Understanding Nonlinear Dynamics.
20. *Holmes:* Introduction to Perturbation Methods.
21. *Curtain/Zwart:* An Introduction to Infinite-Dimensional Linear Systems Theory.
22. *Thomas:* Numerical Partial Differential Equations: Finite Difference Methods.
23. *Taylor:* Partial Differential Equations: Basic Theory.
24. *Merkin:* Introduction to the Theory of Stability.
25. *Naber:* Topology, Geometry, and Gauge Fields: Foundations.
26. *Polderman/Willems:* Introduction to Mathematical Systems Theory: A Behavioral Approach.
27. *Reddy:* Introductory Functional Analysis: with Applications to Boundary-Value Problems and Finite Elements.
28. *Gustafson/Wilcox:* Analytical and Computational Methods of Advanced Engineering Mathematics.
29. *Tveito/Winther:* Introduction to Partial Differential Equations: A Computational Approach.
30. *Gasquet/Witomski:* Fourier Analysis and Applications: Filtering, Numerical Computation, Wavelet.
31. *Bremaud:* Markov Chains: Gibbs Fields and Monte Carlo.
32. *Durran:* Numerical Methods for Wave Equations in Geophysical Fluid Dynamics.

Mark H. Holmes

# Introduction to Perturbation Methods

With 88 Illustrations

Springer

Mark H. Holmes
Department of Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, NY 12180
USA



Mathematics Subject Classifications (1991): 35C20, 34E10, 35B27

Library of Congress Cataloging-in-Publication Data
Holmes, Mark H.
Introduction to perturbation methods / Mark H. Holmes
p. cm. — (Texts in applied mathematics ; 20)
Includes bibliographical references and index.
ISBN-13: 978-1-4612-5349-5 e-ISBN-13: 978-1-4612-5347-1
DOI: 10.1007/978-1-4612-5347-1
1. Perturbation (Mathematics) I. Title. II. Series.
QA871.H74 1995
515'.35—dc20 94-481954

Printed on acid-free paper.



Production managed by Hal Henglein; manufacturing supervised by Jacqui Ashri.
Photocomposed copy prepared using LaTeX.

9 8 7 6 5 4 3 2 (Corrected second printing, 1998)

To Colette, Matthew and Marianna
A small family with big hearts

# Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, had led to the establishment of the series: *Texts in Applied Mathematics* (*TAM*).

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

*TAM* will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences* (*AMS*) series, which will focus on advanced textbooks and research level monographs.

# Preface

First, let me say hello and welcome to the subject of perturbation methods. For those who may be unfamiliar with the topic, the title can be confusing. The first time I became aware of this was during a family reunion when someone asked what I did as a mathematician. This is not an easy question to answer, but I started by describing how a certain segment of the applied mathematics community was interested in problems that arise from physical problems. Examples such as water waves, sound propagation, and the aerodynamics of airplanes were discussed. The difficulty of solving such problems was also described in exaggerated detail. Next came the part about how one generally ends up using a computer to actually find the solution. At this point I editorialized on the limitations of computer solutions and why it is important to derive, if at all possible, accurate approximations of the solution. This lead naturally to the mentioning of asymptotics and perturbation methods. These terms ended the conversation because I was unprepared for their reactions. They were not sure exactly what asymptotics meant, but they were quite perplexed about perturbation methods. I tried, unsuccessfully, to explain what it means, but it was not until sometime later that I realized the difficulty. For them, as in *Webster's Collegiate Dictionary*, the first two meanings for the word perturb are "to disturb greatly in mind (disquiet); to throw into confusion (disorder)." Although a cynic might suggest this is indeed appropriate for the subject, the intent is exactly the opposite. (For a related comment, see Exercise 3.4.1(d).)

In a nutshell, this book serves as an introduction into how to systematically construct an approximation of the solution of a problem that is otherwise intractable. The methods all rely on there being a parameter in the problem that is relatively small. Such a situation is relatively common in applications, and this is one of the reasons that perturbation methods are a cornerstone of applied mathematics. One of the other cornerstones is scientific computing, and it is interesting that the two subjects have grown up together. However, this is not unexpected given their respective capabilities. When using a computer, one is capable of solving problems that are nonlinear, inhomogeneous, and multidimensional. Moreover, it is possible to achieve very high accuracy. The drawbacks are that computer solutions do not provide much insight into the physics of the problem (particularly for those who do not have access to the appropriate software or computer), and there is always the question as to whether or not the computed solution is correct. On the other hand, perturbation methods are also capable of dealing with nonlinear, inhomogeneous, and multidimensional problems (although not to the same extent as computer-generated solutions). The principal objective when using perturbation methods, at least as far as the author is concerned, is to provide a reasonably accurate expression for the solution. By doing this one is able to derive an understanding of the physics of the problem. Also, one can use the result, in conjunction with the original problem, to obtain more efficient numerical procedures for computing the solution.

The methods covered in the text vary widely in their applicability. The first chapter introduces the fundamental ideas underlying asymptotic approximations. This includes their use in constructing approximate solutions of transcendental equations as well as differential equations. In the second chapter, matched asymptotic expansions are used to analyze problems with layers. Chapter 3 describes a method for dealing with problems with more than one time scale. In Chapter 4, the WKB method for analyzing linear singular perturbation problems is developed, while in Chapter 5 a method for dealing with materials containing disparate spatial scales (e.g., microscopic versus macroscopic) is discussed. The last chapter examines the topics of multiple solutions and stability.

The mathematical prerequisites for this text include a basic background in differential equations and advanced calculus. In terms of difficulty, the chapters are written so that the first sections are either elementary or intermediate, while the later sections are somewhat more advanced. Also, the ideas developed in each chapter are applied to a spectrum of problems, including ordinary differential equations, partial differential equations, and difference equations. Scattered through the exercises are applications to integral equations, integro-differential equations, differential-difference equations, and delay equations. What will not be found is an in-depth discussion of the theory underlying the methods. This aspect of the subject is important, and references to the more theoretical work in the area are given in each chapter.

The exercises in each section vary in their complexity. In addition to the more standard textbook problems, an attempt has been made to include problems from the research literature. The latter are intended to provide a window into the wide range of areas that use perturbation methods. Solutions to some of the exercises are available from the author's home page located at http://www.math.rpi.edu/~holmes. Also located there is an errata list. Those who may want to make a contribution to one of these files, or have suggestions about the text, can reach the author at holmes@rpi.edu.

I would like to express my gratitude to the many students who took my course in perturbation methods at Rensselaer. They helped me immeasurably in understanding the subject and provided much needed encouragement to write this book. It is a pleasure to acknowledge the suggestions of Jon Bell, Ash Kapila, and Bob O'Malley, who read early versions of the manuscript. I would also like to thank Julian Cole, who first introduced me to perturbation methods and is still, to this day, showing me what the subject is about.

Troy, New York
August, 1994

Mark H. Holmes

# Contents