

# The Mathematics of Voting and Elections: A Hands-On Approach

Jonathan K. Hodge  
Richard E. Klima

Pro-Nader spots paid for  
Republican Leadership C  
NATION  
Gore receives half a million more  
votes than Bush; loses election

American Idol  
denounced by

2000 *Mathematics Subject Classification*. Primary 91-01; Secondary 91B12.

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**Library of Congress Cataloging-in-Publication Data**

Hodge, Jonathan K., 1980-

The mathematics of voting and elections : a hands-on approach / Jonathan K. Hodge, Richard E. Klima.

p. cm. — (Mathematical world, ISSN 1055-9426 ; v. 22)

Includes bibliographical references and index.

ISBN 0-8218-3798-2 (alk. paper)

1. Voting—Mathematical models. 2. Elections—Mathematical models. 3. Social choice. 4. Game theory. 5. Social sciences—Mathematical models. I. Klima, Richard E. II. Title. III. Series.

JF1001.H63 2005  
324.9'001'5195—dc22

2005041034

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10 9 8 7 6 5 4 3 2 1    10 09 08 07 06 05

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