Undergraduate Texts in Mathematics

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Undergraduate Texts in Mathematics

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(continued after index)

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Mathematical Reflections

In a Room with Many Mirrors

With 134 illustrations

Springer

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To those who showed us
how enjoyable mathematics can be.

Preface

Focusing Your Attention

The purpose of this book is (at least) twofold. First, we want to show you what mathematics *is*, what it is *about*, and how it is *done*—by those who do it successfully. We are, in fact, trying to give effect to what we call, in Section 9.3, our ***basic principle of mathematical instruction***, asserting that "mathematics must be taught so that students comprehend how and why mathematics is done by those who do it successfully."

However, our second purpose is quite as important. We want to attract you—and, through you, future readers—to mathematics. There is general agreement in the (so-called) civilized world that mathematics is important, but only a very small minority of those who make contact with mathematics in their early education would describe it as delightful. We want to correct the false impression of mathematics as a combination of skill and drudgery, and to reinforce for our readers a picture of mathematics as an exciting, stimulating and engrossing activity; as a world of accessible ideas rather than a world of incomprehensible techniques; as an area of continued interest and investigation and not a set of procedures set in stone.

To achieve these two purposes, and to make available to you some good mathematics in the process, we have chosen to present

to you eight topics, organized into the first eight chapters. These topics are drawn both from what is traditionally described as applied mathematics and from what is traditionally described as pure mathematics. On the other hand, these are not topics that are often presented to secondary students of mathematics, undergraduate students of mathematics, or adults wishing to update and upgrade their mathematical competence—and these are the three constituencies we are most anxious to serve. Naturally, we hope that the teachers of the students referred to will enjoy the content of our text and adopt its pedagogical strategy.

Thus, we have chapters on spirals in nature and mathematics, on the designing of quilts, on the modern topic of fractals and the ancient topic of Fibonacci numbers—topics which can be given either an applied or a pure flavor—on Pascal's Triangle and on paper-folding—where geometry, combinatorics, algebra, and number-theory meet—on modular arithmetic, which is a fascinating arithmetic of finite systems, and on infinity itself, that is, on the arithmetic of infinite sets. We have tried to cater to all mathematical tastes; but, of course, we do not claim to be able, through this or any other text, to reach those unfortunate people for whom all mathematical reasoning is utterly distasteful. Pythagoras inscribed on the entrance to his academy, "Let nobody who is ignorant of geometry enter." We might say, at this point, "Let nobody who abhors all mathematics read any further." We see this book as a positive encouragement to those who have already derived some satisfaction from the contact they have had with mathematics. We do not see it as performing a therapeutic function on the "mathophobic"—unless, as is often the case, their mathophobia springs purely from a mistaken view of what mathematics *is*.

You will see that the eight chapters described above are largely independent of one another. We are not at all insisting that you read them in the order in which we have written them. On the other hand, we do also want to stress the *unity* of mathematics, so that there is bound to be a considerable measure of interdependence in the material we present. Cross-referencing will help you to find material from another chapter relevant to the chapter you are currently studying. We should add that Chapter 2 is particularly rich in ideas which play a part in the other chapters of the book.

In our final chapter, Chapter 9, we set out our views of what mathematics *should be* in action, that is, how it should best be done. Naturally, this chapter is quite different in nature from those that precede it. It is not, however, different in tone; we continue the informal, friendly approach which, we very much hope, shows up clearly throughout our text. But we do believe that our readers may find it helpful to have available to them, in easily digestible form, some suggestions as to how to raise their expectations of success in tackling a mathematical exercise. That is our main purpose in including Chapter 9.

We would like to say a few words here about our notation and terminology and our expository conventions. First, we number the items in each chapter separately but as a unity; that is, we start again with each chapter, but, within a chapter, we do not take account of the various sections in our numbering system. Moreover, we have two numbering systems within a chapter; one is the numbering system for theorems, corollaries, examples, and so on, and the other, appearing on the right side of the page, indicates the sequence of displayed formulae of special importance.

Second, we adopt certain conventions and practices in this text to help you to appreciate the significance of the material. From time to time within a chapter we introduce a **BREAK** which gives you the opportunity to test your understanding of the material just presented. At the end of some chapters there is a **FINAL BREAK**, testing your understanding of the entire chapter, which is followed, where appropriate, by a list of **REFERENCES** and the **ANSWERS** to the problems in the final break.[1] As for the references, they are numbered 1, 2, 3, ..., and referred to in the text as [1], [2], [3], ... (You are warned that, in Chapter 2, [3] may be the residue class of the integer 3! The context will make this quite clear and, as we say at the end of the Preface, no notation can be reserved for eternity for one single idea.) From time to time we introduce harder material, which you may prefer to ignore, or to save for a second reading; the beginning and end of such material are marked by a star (⋆) in the left-hand margin and the extent of the difficult material is indicated by a wavy line, also in the left-hand margin,

[1] Our readers are, of course, to be trusted only to consult the answers after attempting the problems.

connecting the two stars. Certain key statements and questions appear displayed in boxes; the purpose of this display is to draw your attention to ideas whose pursuit is going to determine the direction of the subsequent development.

Third, despite our informal approach, we always introduce you to the *correct mathematical term*. In particular, *theorems* are important assertions (not necessarily of a geometrical nature) which are going to be *proved*. *Corollaries* are less important assertions which follow fairly quickly from the previous theorem. *Conjectures*, on the other hand, are hypotheses which may or may not be true, but which we have some rational grounds for believing. It is the standard means of making progress in mathematics to survey what one knows, to make conjectures based on such a survey, to prove these (by logical argument) or disprove them (very often by finding *counterexamples*), to find consequences of the theorems thereby established, and thus to formulate new conjectures (compare Principle 4 in Chapter 9, Section 1). We use LHS and RHS as abbreviations for *left-hand side*, *right-hand side*. We employ the phrase "if and only if" when we are claiming, or proving, that two propositions are *equivalent*. Thus "proposition A if proposition B" means that "B implies A," usually written "$B \Rightarrow A$"; while "proposition A only if proposition B" means[2] that "A implies B" or "$A \Rightarrow B$." The obvious, simple notation for "proposition A if and only if proposition B" is then "$A \Leftrightarrow B$." As a final example, we may use the term *lemma* to refer to an assertion that is to be established for the explicit purpose of providing a crucial step in proving a theorem.[3] Where a proof of an assertion is given rather formally (probably introduced by the word ***Proof***), the end of the proof will be marked by a *tombstone* (□).

It may also be helpful to say a word about the use of the letters of our alphabet, or the Greek alphabet, to represent mathematical

[2] "Proposition A if proposition B" appears sometimes as "B is a *sufficient* condition for A"; while "proposition A only if proposition B" may appear as "B is a *necessary* condition for A." These terms, however, often create difficulties for students.

[3] In fact, the literature of mathematics contains many examples of lemmas that have become more famous than the theorems they were originally designed to prove (e.g., Zorn's Lemma in set theory, Dehn's Lemma in topology, and the Snake Lemma in homological algebra).

concepts like numbers, functions, points, angles, and so on.[4] There are a few—very few—cases in which, throughout mathematical writing, a certain fixed letter is always used for a particular concept; thus, the ratio of the circumference of a circle to its diameter is always π, the base of natural logarithms is always e, the square root of -1 (regarded as a complex number) is always i, a triangle is always Δ, a sum is represented by $\sum$, and so on. However, this does *not* mean that every time these letters appear they refer to the concept indicated above; for example, i may be the subscript for the ith term a_i of a (finite or infinite) sequence of numbers. By the same token we cannot reserve, in the strict sense, any letter throughout an article, much less a book, so that it always refers to the same concept. Of course, different usages of the same letter should be kept far apart, so that confusion is not created; but the reader should remember that any particular usage has a *local* nature—local in place and time. There are far too many ideas in mathematics for each of them to be associated, for all time, with a particular letter of one of four alphabets (small or capital, standard or Greek)—or 40 alphabets, for that matter.

As a final remark in this Preface, it is a pleasure to acknowledge the essential assistance given to us by Kent Pedersen in assembling the Index.

Now you're ready to start, and we hope you have an interesting and enjoyable journey through our text.

Preface to the Second Printing

The authors would like to thank those who drew attention to minor errors in the First Printing.

[4]You will find one use of a letter of the Hebrew alphabet. This is the standard use of the first letter $\aleph$, pronounced "aleph," to represent the cardinality (size) of an infinite set in Chapter 7.

Contents

Preface: Focusing Your Attention vii

1 Going Down the Drain 1

1.1 Constructions 1
1.2 Cobwebs 9
1.3 Consolidation 15
1.4 Fibonacci Strikes 20
1.5 Dénouement 22
Final Break 23
References 23
Answers for Final Break 24

2 A Far Nicer Arithmetic 25

2.1 General Background: What You Already Know . . 25
2.2 Some Special Moduli: Getting Ready for the Fun . 33
2.3 Arithmetic mod p: Some Beautiful Mathematics . 37
2.4 Arithmetic mod Non-primes: The Same But Different 44
2.5 Primes, Codes, and Security 49
2.6 Casting Out 9's and 11's: Tricks of the Trade 54

Final Break 58
Answers for Final Break 59

3 Fibonacci and Lucas Numbers 61

3.1 A Number Trick 61
3.2 The Explanation Begins 63
3.3 Divisibility Properties 72
3.4 The Number Trick Finally Explained 76
3.5 More About Divisibility 78
3.6 A Little Geometry! 81
Final Break 85
References 85
Answers for Final Break 86

4 Paper-Folding and Number Theory 87

4.1 Introduction: What You Can Do With—and Without—Euclidean Tools 87
I *Simple Paper-Folding*
4.2 Going Beyond Euclid: Folding 2-Period Regular Polygons 91
4.3 Folding Numbers 103
4.4 Some Mathematical Tidbits 117
II *General Paper-Folding*
4.5 General Folding Procedures 123
4.6 The Quasi-Order Theorem 128
4.7 Appendix: A Little Solid Geometry 138
Final Break 141
References 141

5 Quilts and Other Real-World Decorative Geometry 143

5.1 Quilts 143
5.2 Variations 153
5.3 Round and Round 162
5.4 Up the Wall 166
Final Break 178
References 182
Answers for Final Break 184

6 Pascal, Euler, Triangles, Windmills, . . . 185

6.1 Introduction: A Chance to Experiment 185
I *Pascal Sets the Scene*
6.2 The Binomial Theorem 187
6.3 The Pascal Triangle and Windmill 196
6.4 The Pascal Flower and the Generalized Star of David 211
II *Euler Takes the Stage*
6.5 Eulerian Numbers and Weighted Sums 217
6.6 Even Deeper Mysteries 238
References 247

7 Hair and Beyond 249

7.1 A Problem with Pigeons, and Related Ideas 249
7.2 The Biggest Number 253
7.3 The Big Infinity 255
7.4 Other Sets of Cardinality $\aleph_0$ 261
7.5 Schröder and Bernstein 268
7.6 Cardinal Arithmetic 269
7.7 Even More Infinities? 270
Final Break 273
References 274
Answers for Final Break 275

8 An Introduction to the Mathematics of Fractal Geometry 277

8.1 Introduction to the Introduction: What's Different About Our Approach 277
8.2 Intuitive Notion of Self-Similarity 280
8.3 The Tent Map and the Logistic Map 290
8.4 Some More Sophisticated Material 298
Final Break 314
References 317
Answers for Final Break 319

9 Some of Our Own Reflections 323

9.1 General Principles 324

9.2 Specific Principles 329
9.3 Appendix: Principles of Mathematical Pedagogy . 336
References 338

Index **339**