American Mathematical Society

Colloquium Publications
Volume 31

Functional Analysis and Semi-Groups

Einar Hille
Ralph S. Phillips

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 47-02;
Secondary 46-02, 47Dxx.

Library of Congress Card Number 58-102
International Standard Book Number 0-8218-1031-6
International Standard Serial Number 0065-9258

Copying and reprinting. Individual readers of this publication, and nonprofit libraries acting for them, are permitted to make fair use of the material, such as to copy a chapter for use in teaching or research. Permission is granted to quote brief passages from this publication in reviews, provided the customary acknowledgment of the source is given.

Republication, systematic copying, or multiple reproduction of any material in this publication (including abstracts) is permitted only under license from the American Mathematical Society. Requests for such permission should be addressed to the Assistant to the Publisher, American Mathematical Society, P.O. Box 6248, Providence, Rhode Island 02940-6248. Requests can also be made by e-mail to `reprint-permission@ams.org`.

Copyright © 1957 by the American Mathematical Society
Printed in the United States of America.

Revised and Expanded Edition 1957.
The American Mathematical Society retains all rights
except those granted to the United States Government.
⊗ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at URL: `http://www.ams.org/`

13 12 11 10 9 8 05 04 03 02 01 00

TABLE OF CONTENTS

PART ONE

FUNCTIONAL ANALYSIS

PAGE

CHAPTER I. ABSTRACT SPACES

1.1. Orientation 2

1. Topological Concepts 2
1.2. Sets.—1.3. Hausdorff spaces.—1.4. Continuous transformations.—1.5. Metric spaces.

2. Additive Spaces 9
1.6. Algebraic systems.—1.7. Additive groups.—1.8. Additive group spaces.

3. Linear Spaces 10
1.9. Linear systems.—1.10. Linear spaces.—1.11. Partially ordered linear systems.—1.12. Banach spaces.

4. Algebraic Spaces 19
1.13. Algebras.—1.14. Algebraic spaces.—1.15. Banach algebras.

CHAPTER II. LINEAR TRANSFORMATIONS

2.1. Orientation 23

1. Additive Transformations 23
2.2. Additive transformations.—2.3. Linear transformations.—2.4. Bounded transformations.—2.5. Subadditive functionals.

2. Linear Functionals 26
2.6. Geometrical aspects.—2.7. Extension of linear functionals.—2.8. The adjoint space.—2.9. The weak topology of $\mathfrak{X}$.—2.10. The weak* topology of $\mathfrak{X}^*$.

3. Linear Transformations 40
2.11. General properties.—2.12. Closed linear transformations.—2.13. Compact linear transformations.

4. Spaces of Endomorphisms 51
2.14. The Banach algebra of endomorphisms.—2.15. The strong and the weak operator topologies.—2.16. Resolvents and spectra.

CHAPTER III. VECTOR-VALUED FUNCTIONS

3.1. Orientation 58

1. Abstract Integration 58
3.2. Some properties of vector-valued functions.—3.3. Riemann-Stieltjes integrals.—3.4. The calculus.—3.5. Measurable functions.—3.6. Countably additive set functions.—3.7. Lebesgue integrals.—3.8. Further properties of the (B)-integral.—3.9. Singular integrals.

2. Complex Function Theory 92
3.10. Holomorphic functions.—3.11. Cauchy's integral.—3.12. Analytic continuation.—3.13. The principle of the maximum.—3.14. The theorem of Vitali.—3.15. Vector-valued functions of several complex variables.

3. Analytic Functions on Vectors to Vectors 109
3.16. (G)-differentiability.—3.17. (F)-differentiability.—3.18. Properties of analytic functions.—3.19. (L)-analyticity.

CHAPTER IV. BANACH ALGEBRAS

4.1. Orientation 117

PAGE

1. General Properties 117
4.2. The unit element.—4.3. Regular elements.—4.4. The kernel.—4.5. Singular elements—4.6. Functions on scalars to the algebra.
2. Spectral Theory 123
4.7. The resolvent.—4.8. The resolvent equations.—4.9. Normed fields.—4.10. Quasi-nilpotent elements.—4.11. Permanent spectral singularities.—4.12. Some properties of (B*)-algebras.
3. Ideal Theory for Commutative Banach Algebras 132
4.13. Ideals.—4.14. Residue-class algebras.—4.15. Representation theory.
4. The Banach Algebra $S(\varphi)$ 141
4.16. The basic properties of $S(\varphi)$.—4.17. Semi-group representations.—4.18. Maximal ideals in $S(\varphi)$.
5. Commutative (A*)-Algebras 151
4.19. Representation theory for commutative (A*)-algebras.—4.20. The algebra $LA(-\infty, \infty)$.—4.21. Some Tauberian theorems.
6. Commutative (B*)-Algebras 157
4.22. Representation theory for commutative (B*)-algebras.—4.23. Operator algebras on Hilbert space.

CHAPTER V. ANALYSIS IN A BANACH ALGEBRA

5.1. Orientation 164
1. The Operational Calculus 164
5.2. Extension of holomorphic scalar functions.—5.3. The spectral mapping theorem. Composite functions.—5.4. The exponential function, the logarithm, and powers.—5.5. Idempotent elements.—5.6. The spectral resolution of an element.—5.7. Spectral theory for compact linear operators.
2. Pseudo-Resolvents 183
5.8. The first resolvent equation.—5.9. Expansion theorems for pseudo-resolvents.—5.10. The second resolvent equation.
3. The Extended Operational Calculus 199
5.11. An operational calculus for closed operators.—5.12. The spectral mapping theorem. Composite functions.—5.13. The spectral resolution of a closed operator.—5.14. Spectral theory for operators with compact resolvents.

CHAPTER VI. LAPLACE INTEGRALS AND BINOMIAL SERIES

6.1. Orientation 212
1. Laplace Transforms 213
6.2. Laplace-Stieltjes integrals.—6.3. Inversion formulas.
2. Functions Holomorphic in a Half-Plane 227
6.4. The classes $H_p(\alpha; \mathfrak{X})$.—6.5. Order relations.—6.6. Representation by Laplace integrals.
3. Binomial Series 231
6.7. Properties of the series.—6.8. Representation and analytic continuation.

PART TWO

BASIC PROPERTIES OF SEMI-GROUPS

CHAPTER VII. SUBADDITIVE FUNCTIONS

7.1. Orientation 237
1. Boundedness and Growth 238
7.2. Elementary properties.—7.3. Semi-modules in E_1 and infinitary solutions.
7.4. Boundedness.—7.5. Negative subadditive functions.—7.6. Rate of growth.

PAGE

2. Continuity and Differentiability 245
7.7. Composition of two-valued subadditive functions.—7.8. Limit functions and continuity.—7.9. Continuity in the mean.—7.10. Moduli of continuity.—7.11. Differentiability.
3. Subadditive Functions in E_n 252
7.12. Positive-homogeneous subadditive functions.—7.13. Boundedness and growth.

CHAPTER VIII. SEMI-MODULES

8.1. Orientation 256
1. Semi-Groups 256
8.2. Abstract semi-groups.—8.3. Transformation semi-groups.—8.4. Examples and illustrations.
2. Euclidean Semi-Modules 261
8.5. The topology of addition.—8.6. Semi-modules on the line.—8.7. Angular semi-modules.
3. Hyperbolical Semi-Groups 271
8.8. Semi-groups of the hyperbolic tangent.—8.9. Suboperative functions.

CHAPTER IX. ADDITION THEOREMS IN A BANACH ALGEBRA

9.1. Introduction.—9.2. Orientation 277
1. Problem A 280
9.3. Measurability implies continuity.—9.4. The exponential solutions.—9.5. Continuous power groups and the logarithm.—9.6. Semi-groups of operators.
2. Problem B 288
9.7. Solutions on Banach spaces.
3. Problem C 291
9.8. Addition theorems.—9.9. Holomorphic solutions.—9.10. Uniqueness of the solution.
4. Problem D 300
9.11. Power series solutions.

CHAPTER X. SEMI-GROUPS IN THE STRONG TOPOLOGY

10.1. Orientation 302
1. One-Parameter Semi-Groups 303
10.2. Measurability and continuity.—10.3. The infinitesimal operator.—10.4. The first exponential formula.—10.5. Continuity at the origin.—10.6. The basic classes of semi-groups.—10.7. Approximation of the identity.
2. Extensions 327
10.8. Problem B in the strong topology.—10.9. The set of generators.—10.10. The n-parameter semi-groups.

CHAPTER XI. GENERATOR AND RESOLVENT

11.1. Orientation 337
1. The Uniform Case 338
11.2. The resolvent.—11.3. Inversion of the resolvent.—11.4. Analytical one-parameter groups of linear transformations.
2. The Strong Case 342
11.5. The resolvent.—11.6. Inversion of the resolvent.—11.7. Special properties.—11.8. The exponential formulas.

CHAPTER XII. GENERATION OF SEMI-GROUPS

12.1. Orientation 356
1. Generation of Semi-Groups Strongly Continuous for $\xi > 0$ 357
12.2. Preliminaries.—12.3. Semi-groups of class (C_0).—12.4. Semi-groups of class $(1, A)$.—12.5. Semi-groups of class (A).

PAGE

2. Generation of Semi-Groups Uniformly Continuous for $\xi > 0$ 377
12.6. Semi-groups of class $(1, A)_u$.—12.7. Semi-groups of class $(A)_\infty$.—12.8. Semi-groups of class $H(\Phi_1, \Phi_2)$.

PART THREE

ADVANCED ANALYTICAL THEORY OF SEMI-GROUPS

CHAPTER XIII. PERTURBATION THEORY

13.1. Orientation 388
1. Perturbation Theory for Semi-Groups of Class $(1, A)$ 389
13.2. Bounded perturbating operators.—13.3. General perturbating operators.—13.4. The perturbed semi-group.—13.5. Classes of infinitesimal generators.—13.6. Cones of infinitestimal generators.
2. Special Results in Perturbation Theory 417
13.7. Semi-groups of class $H(\Phi_1, \Phi_2)$.—13.8. Semi-groups of class $(A)_\infty$.

CHAPTER XIV. ADJOINT THEORY

14.1. Orientation 421
1. Adjoint Operators 422
14.2. The adjoint space.—14.3. Adjoint operators.—14.4. The adjoint semi-group.
2. Adjoint Spaces 429
14.5. The hierarchy of adjoint spaces.—14.6. Reflexive spaces.

CHAPTER XV. OPERATIONAL CALCULUS

15.1. Orientation 434
1. General Theory for Semi-Groups of Class (A) 435
15.2. An elementary calculus.—15.3. The algebra $\mathfrak{S}(\varphi)$.—15.4. The amended calculus.
2. Analytic Dependence for Semi-Groups of Class $(1, A)$ 446
15.5. Analyticity.—15.6. The operator class $\mathfrak{A}_1(\varphi)$.

CHAPTER XVI. SPECTRAL THEORY

16.1. Orientation 453
1. Spectral Mapping Theorems 453
16.2. The commutant of the commutant.—16.3. Representation theory.—16.4. $T(\xi)$ uniformly continuous for $\xi \geqq \gamma > 0$.—16.5. $T(\xi)$ uniformly continuous for $\xi \geqq 0$.
2. Fine Structure 464
16.6. Fine structure mapping theorems.—16.7. The spectrum of $T(\xi; A)$.

CHAPTER XVII. HOLOMORPHIC SEMI-GROUPS

17.1. Orientation 472
1. Domains of Analyticity 473
17.2. Analytic extension of semi-groups.
2. The Structure of Holomorphic Semi-Groups 478
17.3. Growth properties of the norm.—17.4. Holomorphic semi-groups of class (A).—17.5. Semi-groups of class $H(\Phi_1, \Phi_2)$.
3. Semi-Groups and Interpolation Series 488
17.6. Semi-groups holomorphic in a half-plane.—17.7. An embedding theorem.—17.8. Semi-groups with real positive spectra.
4. Boundary Value Problems 498
17.9. Boundary values of a semi-group.—17.10. Groups admitting holomorphic semi-group extensions.

PAGE

CHAPTER XVIII. APPLICATIONS TO ERGODIC THEORY

18.1. Orientation 502

1. Abelian and Tauberian Theorems 503

18.2. Averages.—18.3. Some Tauberian theorems.

2. Ergodic Theory 508

18.4. Ergodicity.—18.5. Weak ergodic theorems.—18.6. Ergodic projections.—18.7. Strong ergodic theorems.—18.8. Uniform ergodic theorems.

PART FOUR

SPECIAL SEMI-GROUPS AND APPLICATIONS

CHAPTER XIX. TRANSLATIONS AND POWERS

19.1. Orientation 530

1. Translations 530

19.2. Translations in $C[0, \infty]$.—19.3. Translations in Lebesgue spaces.—19.4. A nilpotent semi-group.—19.5. Semi-modules and translations.

2. Powers 539

19.6. Power semi-groups.

CHAPTER XX. TRIGONOMETRIC SEMI-GROUPS

20.1. Orientation 543

1. Semi-Groups with Real Parameter 544

20.2. Factor sequences; Lacunary series.—20.3. Semi-groups commuting with translations.—20.4. The case $p = 2$.—20.5. The case $p = 1$.—20.6. Approximation of the identity.

2. Semi-Groups with Complex Parameter 557

20.7. Domains of existence for trigonometric semi-groups.—20.8. Convex semi-module domains of existence.—20.9. Properties of the norm.

CHAPTER XXI. SEMI-GROUPS IN $L_p(-\infty, \infty)$

21.1. Orientation 566

1. Factor Theory 566

21.2. Factor functions for Fourier transforms.—21.3. Factor sequences for Hermitian series.

2. Semi-Groups of the Factor Type 574

21.4. Semi-groups commuting with translations.—21.5. Hermitian semi-groups.

CHAPTER XXII. SEMI-GROUPS IN HILBERT SPACE

22.2. Orientation 583

1. Normal Semi-Groups 583

22.2. Background material.—22.3. Self-adjoint semi-groups.—22.4. Normal semi-groups.—22.5. The unitary group of translations on $L_2(-\infty, \infty)$.

CHAPTER XXIII. MISCELLANEOUS APPLICATIONS

23.1. Orientation 606

1. Abel Summability 607

23.2. Semi-groups connected with Abel summability.—23.3. Some differential operators.

2. Hausdorff Matrices 613

23.4. Hausdorff means.—23.5. Semi-groups in M; the arithmetic of M.

3. The Abstract Cauchy Problem 617

23.6. Cauchy's problem and the principle of scientific determinism.—23.7. The abstract Cauchy problem; uniqueness and non-uniqueness.—23.8. Relations to semi-group theory.—23.9. The abstract Cauchy problem of higher order.

4. Probability Matrices 633

PAGE

23.10. Stochastic processes; the Markoff algebra.—23.11. One-parameter semi-groups in the Markoff algebra.—23.12. Transition operators.

5. Stochastic Processes in E_1 648

23.13. Spatially and temporally homogeneous processes.—23.14. Generation of transition semi-groups.—23.15. Transition semi-groups in E_1^+.

6. Fractional Integration 663

23.16. Riemann-Liouville integrals.—23.17. The integrals of Marcel Riesz.

PART FIVE

EXTENSIONS OF THE THEORY

CHAPTER XXIV. NOTES ON BANACH ALGEBRAS

24.1. Orientation 680

1. Algebras without Unit Element 680

24.2. Reversible elements.—24.3. The dissolvent.—24.4. The dissolvent equations.—24.5. The operational calculus.

2. Ideal Theory 694

24.6. Ideals.—24.7. Residue-class algebras.—24.8. The radical.—24.9. Maximal ideals.—24.10. Maximal ideals and the radical.

3. Representation of (B)-algebras 711

24.11. Linear representation systems.—24.12. Sufficiency of the representations.—24.13. Functions of maximal ideals.

4. A Structure Theorem 718

24.14. The structure of normed groups and semi-groups.

CHAPTER XXV. LIE SEMI-GROUPS

25.1. Orientation 726

1. Semi-Groups in a (B)-Space 727

25.2. Basic assumptions.—25.3. The canonical function.—25.4. The canonical orbits.

2. Parametric Transformation Semi-Groups 741

25.5. Further assumptions; suboperative functions.—25.6. Uniform continuity of $T(p)$.—25.7. Strong continuity of $T(p)$.

3. The Lie Theory 748

25.8. The structural constants.—25.9. The infinitesimal generators.—25.10. The fundamental theorems.

CHAPTER XXVI. FUNCTIONS ON VECTORS TO VECTORS

26.1. Orientation 760

1. Differentiable Functions 760

26.2. Multilinear forms and polynomials.—26.3. Differentiability.—26.4. The Taylor series of an (L)-analytic function.

2. Power Series 770

26.5. (G)-power series.—26.6. (F)-power series.—26.7. A theorem on analyticity.

BIBLIOGRAPHY 780

INDEX 796