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Michel Hénon

# Generating Families in the Restricted Three-Body Problem

II. Quantitative Study of Bifurcations



Springer

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# Preface

In a previous volume (Hénon 1997, hereafter called volume I), the study of generating families in the restricted three-body problem was initiated. (We recall that generating families are defined as the limits of families of periodic orbits for  $\mu \rightarrow 0$ .) The main problem was found to lie in the determination of the junctions between the branches at a bifurcation orbit, where two or more families of generating orbits intersect. A partial solution to this problem was given by the use of invariants: symmetries and sides of passage. Many simple bifurcations can be solved in this way. In particular, the evolution of the nine natural families of periodic orbits can be described almost completely. However, as the bifurcations become more complex, i.e. when the number of families passing through the bifurcation orbit increases, the method fails.

This volume describes another approach to the problem, consisting of a detailed, quantitative analysis of the families in the vicinity of a bifurcation orbit. This requires more work than the qualitative approach used in Vol. I. However, it has the advantage of allowing us, in principle at least, to determine in all cases how the branches are joined. In fact it gives more than that: we will see that, in almost all cases, the first-order asymptotic approximation of the families in the neighbourhood of the bifurcation can be derived. This allows, in particular, a quantitative comparison with numerically found families.

Chapter 11 deals with the relevant definitions and general equations. The quantitative study of bifurcations of type 1 is described in Chaps. 12–16. The analysis of type 2 is more involved; it is described in Chaps. 17–23. Type 3 is even more complex; its analysis had not yet been completed at the time of writing.

As was the case for the previous volume, this work is sometimes lacking in mathematical rigor; there is certainly much room for improvement. However, a number of factors lead me to believe that the results are correct: agreement with the results of the qualitative analysis of Vol. I; agreement with numerical computations; internal consistency; and, simple intuition.

My thanks go to Larry Perko, who read a draft version of this volume and made many helpful comments and suggestions. I also thank Alexander Bruno for many discussions by e-mail, and for sending an english translation of parts of his new book in advance of publication.

Nice, March 2001

*Michel Hénon*

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