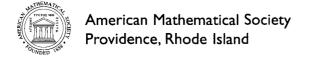
## Differential Geometry, Lie Groups, and Symmetric Spaces

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## **CONTENTS**

Pref	ACE									xiii
PREFA	ACE TO THE 2001 PRINTING	-								xvii
Suga	GESTIONS TO THE READER									xix
SEQU	EL TO THE PRESENT VOLUME									xxi
GROU	JPS AND GEOMETRIC ANALYSIS CONTENTS									xxiii
GEON	METRIC ANALYSIS ON SYMMETRIC SPACES C	ONT	ENTS							xxv
	С	HAI	PTE	R I						
	Elementary D	iffe	rer	ntial	Ge	om	etry	•		
1.	Manifolds				,					2
2.	Tensor Fields									8
	1. Vector Fields and 1-Forms									8
	2. Tensor Algebra									13
	3. The Grassman Algebra .				•					17
	4. Exterior Differentiation .									19
3.	Mappings									22
	1. The Interpretation of the Ja									22
	2. Transformation of Vector Fi	ields								24
	3. Effect on Differential Forms									25
4.	Affine Connections									26
5.										28
6.	The Exponential Mapping .									32
7.										40
8.										43
9.	The Riemannian Connection .									47
10.	Complete Riemannian Manifolds									55
11.	Isometries									60
12.										64
13.	Riemannian Manifolds of Negativ		ırva	ture						70
14.	Totally Geodesic Submanifolds									78
15.	Appendix									82
										82
	2. Mappings of Constant Rank									86
	Exercises and Further Results									88
										95
	CI	HAF	PTE	R II						
	Lie Groups	s an	nd L	.ie A	Alge	bra	S			
1.	The Exponential Mapping .									98
	1. The Lie Algebra of a Lie Gr	roup								98

X CONTENTS

	2. The Universal Enveloping Algebra									100
	3. Left Invariant Affine Connections .							,		102
	4. Taylor's Formula and the Differents	ial d	f the	Exponer	ntial	Map	ping			104
2.	Lie Subgroups and Subalgebras									112
3.	Lie Transformation Groups									120
4.	Coset Spaces and Homogeneous Spaces									123
5.	The Adjoint Group									126
6.	Semisimple Lie Groups									131
7.	Invariant Differential Forms									135
8.	Perspectives									144
	Perspectives									147
	Notes									153
				•						
	CHAPTI	ER	Ш							
	Structure of Semisir	np	le Li	ie Alge	bra	ıs				
1.	Preliminaries				_					155
2.	Theorems of Lie and Engel						•	•		158
3.			•		•	•	•	•	•	162
4.	Cartan Subalgebras			•	•			•	•	165
5.	Significance of the Root Pattern							•	•	171
6.	<del>-</del>							•	•	178
7.			•	•	•		•	•	٠	182
8.	Cartan Decompositions			•	•		•	•	•	186
٥.	Examples. The Complex Classical Lie Al	igeo	ras .	•	•	•			•	191
	Exercises and Further Results Notes			•	• .			•	٠	191
	Notes		• •	•	•	•	•	•	•	170
	CHAPTE	ER	IV							
	Symmetric	c S	расе	es						
1.	Affine Locally Symmetric Spaces				-					198
2.	Groups of Isometries				•			•	•	201
3.	Riemannian Globally Symmetric Spaces		•			:		•	•	205
<i>3</i> .	The Exponential Mapping and the Curv				•			•	•	214
<del>5</del> .	Locally and Globally Symmetric Spaces	atu	16 .	•				•	•	218
									•	223
6.	Compact Lie Groups					•	•	•	٠	224
7.	Totally Geodesic Submanifolds. Lie Trip				•	•	•	•	•	224
	Exercises and Further Results		•		•	•	•	•	•	227
	Notes		•		•	•	•	•	•	221
	CHAPT	ER	٧							
	Decomposition of S	ym	me	tric Sp	ace	s				
1.	Orthogonal Symmetric Lie Algebras .			•						229
2.			•	•	•	•	•	•	•	235
2. 3.	The Duality				•	•	•	•	•	241
3. 4.					•	•	•	•	•	243
4.	Symmetric Spaces with Semisimple Grou	ıps (	01 120	inethies	•	•	•	•	•	243

CONTENTS	Xì

_												
5.	Notational Conventions	•	٠	•	•	٠	•	•	•	•	•	244
6.	Rank of Symmetric Spaces	٠	•	•	•	٠	•	•	•	•	•	
	Exercises and Further Results	٠	•	•	•	•	•	•	•	•	•	249
	Notes	•	•	•	•	•	•	•	•	•	•	251
	CI	HAF	TEF	₹ VI								
	Symmetric Spaces	s of	the	No	nco	mp	act	Tyl	рe			
1.	Decomposition of a Semisimple L	ie G	roup									252
2.	Maximal Compact Subgroups and											256
3.	The Iwasawa Decomposition .			_								257
4.	Nilpotent Lie Groups											264
5.	Global Decompositions											270
6.	The Complex Case											273
	Exercises and Further Results						-					275
			-									279
		•	•	•	•	•	•	•	·	•	•	,
	Cl	HAP	TER	. VII								
	Symmetric Space	ces (	of t	he (	Com	pac	t T	уре				
1.	The Contrast between the Comp	act 1	Гvne	and	the	None	com:	nact	Type	P.		281
2.	The Weyl Group and the Restrict								- ) -	•	•	283
3.	Conjugate Points. Singular Points.					•	•	·	•	•	•	293
4.	Applications to Compact Groups					·	•	·	:	•	•	297
5.	Control over the Singular Set .					•	•	•	•	•	•	303
6.	The Fundamental Group and the					Ċ		•	•	•	•	307
7.	The Affine Weyl Group				:		:	•	•	•	•	314
8.	Application to the Symmetric Sp							•	•	•	:	318
9.	Classification of Locally Isometri						:		•	•	•	325
0.	Geometry of $U/K$ . Symmetric Sp								•	•	•	327
1.	Shortest Geodesics and Minimal								•	•	•	334
2.	Appendix. Results from Dimensio		-			-			•	•	•	344
	Exercises and Further Results			, .	•	•	•	•	•	•	•	347
	Notes	•	•	•	•	•	•	•	•	•	•	350
	110103	•	•	•	•	•	•	•	•	•	•	330
	CH	IAP	TER	VII								
	Hermitian	Syı	mm	etri	c Sp	oace	es.					
1.	Almost Complex Manifolds .											352
2.	Complex Tensor Fields. The Ricc											356
3.	Bounded Domains. The Kernel											364
4.	Hermitian Symmetric Spaces of th	e Co	mpa	ct T	ype a	nd tl	ne N	onco	mpa	ct Ty	тре	372
5.	Irreducible Orthogonal Symmetric								•		٠.	377
6.	Irreducible Hermitian Symmetric		_									381
7.	Bounded Symmetric Domains											382
	Exercises and Further Results											396
	Notes											400

## CHAPTER IX

Structure (	of Sem	isimple	Lie	Group	ps
-------------	--------	---------	-----	-------	----

1.	Cartan, Iwasawa, and Bruhat	Deco	ompo	osition	ns								401
2.	The Rank-One Reduction		. •										407
3.	The $SU(2, 1)$ Reduction .												409
4.													418
5.	Automorphisms												421
6.	The Multiplicities											•	428
7.	Jordan Decompositions .										•	•	430
•	Evercises and Further Recult	ro.			•	•	•	•	•	•	•	•	434
	Exercises and Further Result Notes		•	•	•	•	•	•	•	•	•	•	436
	110100	•	•	•	•	•	•	•	•	•	•	•	730
		Cl	HAF	TER	Χ								
	The Classifica				-			lgeb	ras				
	and o	f Sy	mı	netr	ic S	Spac	es						
1.	Reduction of the Problem												438
2.	The Classical Groups and Th												444
	1. Some Matrix Groups a												444
	2. Connectivity Properties												447
	3. The Involutive Automor	rphis	ms o	f the	Ćlas	ssical	Con	npact	Lie	Alge	bras		451
3、	Root Systems												455
	1. Generalities												455
	2. Reduced Root Systems												459
	3. Classification of Reduce								hs a	nd I	Dynk:	in	
	Diagrams												461
	4. The Nonreduced Root S	vster	ns										474
	5. The Highest Root .												475
	6. Outer Automorphisms a	nd t	he C	overi	ng In	ıdex							478
4.	The Classification of Simple												481
5.	Automorphisms of Finite Ord												490
6.	The Classifications												515
•	1. The Simple Lie Algebra	25 07	er (	Cand	The	ir C	ombo	et R	eal l	orm	s. Ti	ne.	
	Irreducible Riemannian (												515
	2. The Real Forms of Sim												
	Globally Symmetric Spo												517
	3. Irreducible Hermitian S									Ţ		•	518
	4. Coincidences between Di									•	•	•	518
	Exercises and Further Result				. <i>Бр</i>						•	•	520
	Notes			•	•	•	•	•	•	•	•		535
	Notes	•	•	•	•	•	•	•	•	•	•	•	333
Solu	UTIONS TO EXERCISES												538
Som:	e Details												586
Віві	LIOGRAPHY												599
	OF NOTATIONAL CONVENTIONS												629
	BOLS FREQUENTLY USED .												632
	EX												635
Revi	EWS FOR THE FIRST EDITION -						•						641