Lecture Notes in Mathematics

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Semi-Classical Analysis for the Schrödinger Operator and Applications

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§0 INTRODUCTION.

This course falls into two different parts. The first part (Sections 1-5) is the written version of ten lectures I gave in Nankai University in October 1985. It can be seen as an introduction to my work with J. Sjöstrand ([HE-SJ]$_{1-6}$). My purpose was to give in a simpler situation a relatively self-contained presentation of the tunneling effect. In fact, we have tried to refer only to the two basic books of Reed-Simon [RE-SI] and Abraham-Marsden. [AB-MA] (see also Abraham-Robin for the theory of the stable manifolds). The material presented here comes essentially from [HE-SJ]$_1$ but we have also used improvements that we found later in [HE-SJ]$_{2-6}$ and the proof presented here is, at least in the form, partly different (particularly for the B.K.W construction, where we present a simpler method, less general, but perhaps easier to understand for non-specialists in microlocal analysis).

Almost two years later, in June 1987, I was asked to complete these notes to permit a publication as a volume of the Springer Lecture Notes (Nankai Subseries). During these two years, we had applied these techniques, in collaboration with J. Sjöstrand or through students, to many other problems where the tunneling effect plays an important role: resonances, Schrödinger with periodic potential, Schrödinger with magnetic fields, etc... but it is probably too early to write a definite book on the subject. At the same time, a very good book on the Schrödinger operator by Cycon-Froese-Kirsch-Simon [C.F.K.S] has appeared. We have therefore chosen to present in § 6 and § 7 subjects which are complementary to this book and which are natural applications of the theory developed in the first 5 sections. This book is organized as follows.

In § 1, we present a brief survey of semi-classical mechanics and recall basic facts on the Schrödinger operator. This material is more developed in the recent book by D. Robert [Ro] which we recommend to the reader interested in pseudodifferential techniques.
§ 2 is concerned with the B.K.W construction at the bottom. In § 3, we study the decay of the eigenfunctions in the spirit of Agmon [AG]. These results were developed in the semi-classical context by B. Simon [SI]$_{2-4}$ and B. Helffer-J. Sjöstrand [HE-SJ]$_{1-9}$.
§ 4 is concerned with the interaction between different wells. This is a mathematical version of the well-known L.C.A.O method used by chemists.
In § 5, we present briefly the application to Witten's proof of the Morse inequalities [WIT]. There is an intersection with one chapter of the book [C.F.K.S] but we have tried to go a little further using the techniques of sections 2,3,4, however we still remain far from the best results (related to the method of instantons) obtained in [HE-SJ]$_4$. In § 6, we study the asymptotic behavior of the first band of

the Schrödinger operator with periodic potentials and present results obtained by
B.Simon $[SI]_6$ and A. Outassourt [OU].

§ 7 is devoted to the study of some classical problems on the Schrödinger operator
with magnetic fields: criteria for the compactness of the resolvent (after Helffer-
Mohamed) [HE-MO]), multiplicity of the first eigenvalue (after Avron-Herbst-Simon
[A-H-S], Lavine-O'Carroll [LA, O'CA], Helffer-Sjöstrand $[HE-SJ]_{10}$ and Helffer
[HE]), effect of the flux of the magnetic field $[HE]_3$). The study of these problems
is only beginning and we just give a flavor of some of the problems (see also the
chapter in [C.F.K.S] devoted to these questions).

I have many people to thank at the end of this introduction. First of all the
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I want also to thank J. Sjöstrand and D. Robert with whom I have collaborated in
this field, M. Dauge who read a part of the text and C. Brunet and M. Coignac who
typed the manuscript.
For the reader who does not appreciate my poor English, let me mention in closing
that there exists a Chinese version of this course, written up by Professor
Chi Min-yu and his students.

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