

A First Course in Logic

An introduction to model theory, proof theory,
computability, and complexity

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