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Stability of Motion

Translated by

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Preface

The theory of the stability of motion has gained increasing significance in the last decades as is apparent from the large number of publications on the subject. A considerable part of this work is concerned with practical problems, especially problems from the area of controls and servo-mechanisms, and concrete problems from engineering were the ones which first gave the decisive impetus for the expansion and modern development of stability theory.

In comparison with the many single publications, which are numbered in the thousands, the number of books on stability theory, and especially books not written in Russian, is extraordinarily small. Books which give the student a complete introduction into the topic and which simultaneously familiarize him with the newer results of the theory and their applications to practical questions are completely lacking. I hope that the book which I hereby present will to some extent do justice to this double task. I have endeavored to treat stability theory as a mathematical discipline, to characterize its methods, and to prove its theorems rigorously and completely as mathematical theorems. Still I always strove to make reference to applications, to illustrate the arguments with examples, and to stress the interaction between theory and practice.

The mathematical preparation of the reader should consist of about two to three years of university mathematics. Here and there a few fundamental concepts of the theory of metric spaces are needed, but I have formulated the arguments in such a way that the reader can usually find an interpretation in n-dimensional Euclidean space. On the whole I limited the selection of materials mainly to the stability of motions in Euclidean space, particularly since the majority of applications are concerned with such motions. But I have stated the basic definitions of stability and proved a number of criteria in a general form, and pointed out take-off points for further investigations, as for instance in the theory of differential equations and difference equations.

Even when limited to Euclidean space (*i.e.* to common differential and difference equations) a complete presentation of the field is not possible in an introduction. Many fine isolated results had to be left out. But I also had to omit several larger topics. Among them was the stability of random processes (*cf.* the remark in sec. 36) as well as the method of "harmonic balance" in the section on periodic motions, which although mathematically suspect is indispensible to the practical man. To include the work done to give a rigorous foundation to such methods (BOGOLYUBOV and MITROPOLSKII in Russia, CESARI, HALE, and co-workers in the USA) would have gone beyond the limits of this book.

It is only natural that the section on periodic motions is rather short compared to the other sections. After all, the subject matter of stability theory involves primarily assertions about the stability of the equilibrium, whereas the numerous contributions to periodic motions are mainly concerned with existence questions.

It was not necessary to treat in more detail individual investigations on second-order differential equations since only quite recently an excellent monograph by REISSIG, SANSONE, and CONTI has appeared, which reports on the newest developments.

As I intended to write a text book and not a handbook, the bibliography is by no means complete. It comprises those publications which I actually used (in the text I frequently referred to sources) and several works of interest for further study.

In preparing the work I received very valuable help from many sources. Dr. KAPPEL read the manuscript and the galley-proofs and sketched the figures. Dr. W. MÜLLER, Prof. REISSIG, Dr. STETTNER, and Dr. INGE TROCH read the galley-proofs and made many critical remarks. I wish to express my sincerest gratitude to all the above. I especially thank Prof. ARNE P. BAARTZ for translating the German manuscript into English and also for making a number of useful suggestions.

A major part of the manuscript was written during my stay at the Mathematics Research Center, Madison, Wis. I am very much obliged to its Directors, Prof. R. E. LANGER and Prof. J. B. ROSSER, for having arranged that stay. Finally, I wish to acknowledge the cooperation of the Springer-Verlag and to appreciate the consideration given to this book by the Editors of the Yellow Series.

Graz, July 1967

WOLFGANG HAHN

Contents

Notations and Formulas

1. The Definition Symbol. The symbol $:=$ or $=:$ defines the variable standing next to the colon. For example, $c := f(a, b)$. c is being introduced, the right side is known.

2. End of Proof. In case of a longer proof, the end of the proof is indicated by the symbol $\bullet$. It corresponds to the classical q.e.d. Sometimes, it denotes the end of the discussion of an example.

3. Vectors. Vectors in n-dimensional Euclidean space R_n are denoted by lower case Latin, and occasionally Greek letters in semi-bold face. Their components have indices. All vectors are column vectors

$$\boldsymbol{x} = \text{col}\,(x_1, \ldots, x_n)\,.$$

$\boldsymbol{x}^T$ is the transpose of $\boldsymbol{x}$ and thus is a row vector; $\boldsymbol{x}^T\boldsymbol{y}$ is the inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$. The norm $|\boldsymbol{x}|$ is the Euclidean norm

$$|\boldsymbol{x}|^2 := x_1^2 + \cdots + x_n^2 = \boldsymbol{x}^T\boldsymbol{x}\,.$$

The zero vector $\text{col}\,(0, \ldots, 0)$ is simply denoted by 0. The inequality

$$|\boldsymbol{x}| < a$$

defines an open ball in R_n with radius a and center at the origin; it is denoted by K_a. The "half cylinder" defined by $|\boldsymbol{x}| < a, t \geq t_0$, is denoted by K_{a,t_0}.

4. Matrices[1] are denoted by capital Latin letters. Their elements are represented by lower case Latin letters with double indices, *e.g.*

$$A = (a_{ik}), \quad i, k = 1, 2, \ldots, n\,.$$

The determinant of A is denoted by $\det A$, its trace by $\text{Tr}\,A$,

$$\text{Tr}\,A := \sum_{i=1}^{n} a_{ii}\,.$$

If $a_{ij} = 0$, $i \neq j$, we write

$$A = \text{diag}\,(a_{11}, \ldots, a_{nn}).$$

A^T is the transpose of the matrix A, A^I is the inverse of A in case $\det A \neq 0$. E is the unit matrix. Bilinear forms are written in vector notation

$$\sum_{i,k=1}^{n} a_{ik}\,x_i\,y_k =: \boldsymbol{x}^T A\,\boldsymbol{y}\,.$$

[1]) *Cf.*, for instance, Bellman [2], Schmeidler [1].

If the quadratic form $x^T Bx$, B symmetric, is positive definite we write $B > 0$.
The norm of A is the square root of the largest characteristic root of $A^T A$, or equivalently,

$$||A|| := \sup \left\{ \frac{|A\,x|}{|x|} \quad x \in R_n \right\}.$$

5. Differentiation. For a scalar function f depending on a variable vector x (a mapping of a domain in R_n into the real line) we write

$$f_x = \frac{\partial f}{\partial x} := \operatorname{grad} f = \operatorname{col}\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

The derivative of a vector $g(x)$ with respect to x is the Jacobian or functional matrix

$$\frac{\partial g}{\partial x} := \left(\frac{\partial g_i}{\partial x_k}\right), \quad i = 1, 2, \ldots, m; \; k = 1, 2, \ldots, n.$$

The derivative with respect to the "time" variable t is denoted by a raised dot, $\dot{x} := \frac{dx}{dt}$.

The differential equation

$$\dot{x} = f(x, t)$$

replaces the n scalar differential equations

$$\dot{x}_i = f_i(x_1, \ldots, x_n, t), \quad i = 1, 2, \ldots, n.$$

In general we shall assume without further mention that the equation possesses solutions in a certain neighborhood K_h of the origin which are uniquely determined by the *initial time* t_0 and the *initial point* x_0. The solution determined by t_0 and x_0 is denoted by $p(t, x_0, t_0)$, so that $p(t_0, x_0, t_0) = x_0$.

Uniqueness is assured if $f(x, t)$ satisfies a Lipschitz condition ($f \in C_0$), *i.e.* if there exists a number L such that

$$|f(\bar{x}, t) - f(x, t)| < L\,|\bar{x} - x|$$

for all $\bar{x}, x \in K_h$, $t_0 \le t \le t_1$. In that case there is an estimate

$$|\bar{x}_0 - x_0|\, e^{-nL|t-t_0|} \le |p(t, \bar{x}_0, t_0) - p(t, x_0, t_0)| \le |\bar{x}_0 - x_0|\, e^{+nL|t-t_0|}.$$

If $f(x, t)$ satisfies this condition with the same L for all $\bar{x}$, x in the closed ball $|x| \le h$ and for all $t \ge t_0$, we say that f satisfies a Lipschitz condition with a uniform Lipschitz constant ($f \in \bar{C}_0$) (*cf.* also CODDINGTON and LEVINSON [1], KAMKE [1], among others).

6. Point Sets. If G is a point set in R_n, $\bar{G}$ denotes the closure, $b(G)$ its boundary. If $G_2 \subset G_1$, $G_1 \setminus G_2$ denotes the difference.