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Solving Ordinary Differential Equations I

Nonstiff Problems

Second Revised Edition
With 135 Figures

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