

Lecture Notes in Mathematics

A collection of informal reports and seminars
Edited by A. Dold, Heidelberg and B. Eckmann, Zürich

143

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Cohomological Topics
in Group Theory



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PREFACE

These notes are based on lectures that I have given at various times during the last four years and at various places, but mainly at Queen Mary College, London. Chapters 1 to 7 have been in circulation as a volume in the Queen Mary College Mathematics Notes since the autumn of 1967. They are reproduced here unchanged except for the addition of some bibliographical material and the correction of some minor errors.

Chapter 8 is an attempt at a reasonably complete survey of the subject of finite cohomological dimension. I have included proofs of everything that is not readily accessible in the literature.

Chapters 9 and 11 contain an account of a kind of globalised extension theory which I believe to be new. A survey of some of the results has appeared in volume 2 of "Category theory, homology theory and their applications", Springer Lecture Notes, no.92 (1969). The basic machinery of extension categories for arbitrary groups is given in chapter 9. Then in chapter 11 we focus attention exclusively on finite groups and primarily on the structure of minimal projective extensions. Chapter 10 is purely auxiliary and merely sets out some cohomological facts needed in chapter 11.

My aim in these lectures was to present cohomology as a tool for the study of groups. In this respect they differ basically from other available accounts of group cohomology in

all of which the theory is developed with an eye on arithmetical applications. Our subject here is group theory with a cohomological flavour.

It should be stressed that there is no pretence whatsoever at completeness. In fact, the general homological machinery is kept to the bare minimum needed for the topics at hand. It follows - inevitably - that many important features are barely mentioned; and some not at all.

The audiences were not assumed to know anything about homological algebra except the most rudimentary facts. A little more knowledge of group theory was presupposed, but nothing at all sophisticated. Full references to all non-trivial or non-standard results are always given.

There is a list of the most frequently quoted books immediately following this preface. Each chapter ends with a list of all articles and books mentioned in that chapter and reference numbers refer to that list at the end of the chapter where they occur.

I was fortunate to have perceptive audiences who frequently saved me from errors and obscurities. My thanks go to all who participated and in particular to D. Cohen, I. Kaplansky, D. Knudson, A. Learner, H. Mochizuki, G. Rinehart, W. Vasconcelos and B. Wehrfritz. I owe a special debt of gratitude to Urs Stambach for his careful and critical reading of large sections of these notes.

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BOOK LIST

The following books are usually referred to by their author's name only.

Burnside, W.: The theory of groups of finite order, Cambridge, 2nd edition, 1911 (Chelsea 1958).

Cartan, H. and Eilenberg, S.: Homological algebra, Princeton 1956.

Curtis, C.W. and Reiner, I.: Representation theory of finite groups and associative algebras, Interscience, 1962.

Hall, P.: Nilpotent groups, Notes of lectures at the Canadian Mathematical Congress, Univ. of Alberta, 1957. (Reprinted: Queen Mary College Mathematics Notes, 1969).

Huppert, B.: Endliche Gruppen I, Springer, 1967.

Lang, S.: Rapport sur la cohomologie des groupes, Benjamin, 1966.

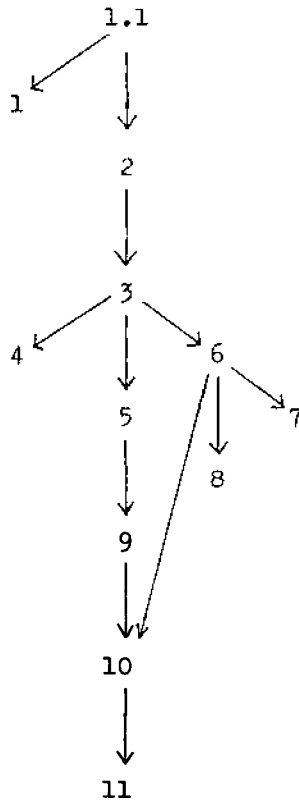
Rotman, J.: The theory of groups: an introduction, Allyn and Beacon, 1965.

Schenkman, E.: Group theory, van Nostrand, 1965.

Scott, W.R.: Group theory, Prentice-Hall, 1964.

Serre, J.-P.: Corps Locaux, Hermann, 1962.

UNGEFÄHRER LEITFADEN.



SOME NOTATION AND TERMINOLOGY

Let G be a group.

If S is a subset of a G -group M (p.1), $\langle G\langle S \rangle$ is the G -subgroup generated by S .

We write $\langle 1\langle S \rangle = \langle S \rangle =$ subgroup generated by S .

$\text{Fr}_G(M) =$ G -Frattini group of M (§7.1).

$d_G(M) =$ minimum number of G -generators of M (§7.1).

$N_G(A) =$ normalizer of A in G .

$C_G(A) =$ centralizer of A in G .

$|G|, |x| =$ order of G, x .

A complete set of representatives of the (right) cosets of A in G is called a (right) transversal of A in G .

$G[M = M]G =$ split extension with kernel M and complement G (§1.1).

$\Pi =$ product (Cartesian product).

$\coprod =$ coproduct. This is the direct sum in Mod_G , the category of G -modules. It is $*$, the free product, in the category of groups.

$[a, b] = a^{-1}b^{-1}ab =$ commutator of a by b .

If A, B are subsets of G , $[A, B] = \langle [a, b] \mid a \in A, b \in B \rangle$.

If H, K are subgroups of G and $K \triangleleft H$ (normal), H/K is a factor of G .

If $[H, G] \leq K$, the factor is called central.

A finite series is a family of subgroups $(S_i; 0 \leq i \leq m)$, where

$$S_i \triangleleft S_{i+1}.$$

If all factors are central, the series is called a central series.

If G has a finite central series from 1 to G (i.e., $S_0 = 1$ and $S_m = G$), then G is called nilpotent.

If G has a finite series from 1 to G with all factors abelian (cyclic), then G is called soluble (polycyclic).

$h(G)$ = Hirsch number of the locally polycyclic G (§8.8).

If $\zeta_0(G) = 1$, $\zeta_1(G)$ = centre of G , and $\zeta_{k+1}(G)$ is the unique subgroup so that $\zeta_{k+1}(G)/\zeta_k(G) = \zeta_1(G/\zeta_k(G))$, then $(\zeta_i(G); i \geq 0)$ is called the upper central series of G .

If G is nilpotent and $\zeta_{c-1}(G) < \zeta_c(G) = G$, then c is the class of G .

If $G_1 = G$, $G_{k+1} = [G_k, G]$, then $(G_i; i \geq 1)$ is called the lower central series of G .

If $G^{(0)} = G$, $G' = [G, G]$ ($= G_2$) and $G^{(m+1)} = (G^{(m)})'$, then $(G^{(i)}; i \geq 0)$ is called the derived series of G .