Lecture Notes in Mathematics

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Karl W. Gruenberg
Queen Mary College, London

Cohomological Topics in Group Theory

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PREFACE

These notes are based on lectures that I have given at various times during the last four years and at various places, but mainly at Queen Mary College, London. Chapters 1 to 7 have been in circulation as a volume in the Queen Mary College Mathematics Notes since the autumn of 1967. They are reproduced here unchanged except for the addition of some bibliographical material and the correction of some minor errors.

Chapter 8 is an attempt at a reasonably complete survey of the subject of finite cohomological dimension. I have included proofs of everything that is not readily accessible in the literature.

Chapters 9 and 11 contain an account of a kind of globalised extension theory which I believe to be new. A survey of some of the results has appeared in volume 2 of "Category theory, homology theory and their applications", Springer Lecture Notes, no.92 (1969). The basic machinery of extension categories for arbitrary groups is given in chapter 9. Then in chapter 11 we focus attention exclusively on finite groups and primarily on the structure of minimal projective extensions. Chapter 10 is purely auxiliary and merely sets out some cohomological facts needed in chapter 11.

My aim in these lectures was to present cohomology as a tool for the study of groups. In this respect they differ basically from other available accounts of group cohomology in all of which the theory is developed with an eye on arithmetical applications. Our subject here is group theory with a cohomological flavour.

It should be stressed that there is no pretence whatsoever at completeness. In fact, the general homological machinery is kept to the bare minimum needed for the topics at hand. It follows - inevitably - that many important features are barely mentioned; and some not at all.

The audiences were not assumed to know anything about homological algebra except the most rudimentary facts. A little more knowledge of group theory was presupposed, but nothing at all sophisticated. Full references to all non-trivial or non-standard results are always given.

There is a list of the most frequently quoted books immediately following this preface. Each chapter ends with a list of all articles and books mentioned in that chapter and reference numbers refer to that list at the end of the chapter where they occur.

I was fortunate to have perceptive audiences who frequently saved me from errors and obscurities. My thanks go to all who participated and in particular to D. Cohen, I. Kaplansky, D. Knudson, A. Learner, H. Mochizuki, G. Rinehart, W. Vasconcelos and B. Wehrfritz. I owe a special debt of gratitude to Urs Stammbach for his careful and critical reading of large sections of these notes.

I am also grateful to Cornell University, the University of Oregon, the University of British Columbia and the Eidgen. Tech. Hochschule, Zürich, for financial assistance at various stages of this work.

The notes were typed by Mrs. Esther Monroe and Miss Valerie Kinsella and I thank them both for their enormous patience with me and their excellent work.

Queen Mary College,
London,
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BOOK LIST

The following books are usually referred to by their author's name only.

Burnside, W.: The theory of groups of finite order, Cambridge, 2nd edition, 1911 (Chelsea 1958).

Cartan, H. and Eilenberg, S.: Homological algebra, Princeton 1956.

Curtis, C.W. and Reiner, I.: Representation theory of finite groups and associative algebras, Interscience, 1962.

Hall, P.: Nilpotent groups, Notes of lectures at the Canadian Mathematical Congress, Univ. of Alberta, 1957. (Reprinted: Queen Mary College Mathematics Notes, 1969).

Huppert, B.: Endliche Gruppen I, Springer, 1967.

Lang, S.: Rapport sur la cohomologie des groupes, Benjamin, 1966.

Rotman, J.: The theory of groups: an introduction, Allyn and Beacon, 1965.

Schenkman, E.: Group theory, van Nostrand, 1965.

Scott, W.R.: Group theory, Prentice-Hall, 1964.

Serre, J.-P.: Corps Locaux, Hermann, 1962.

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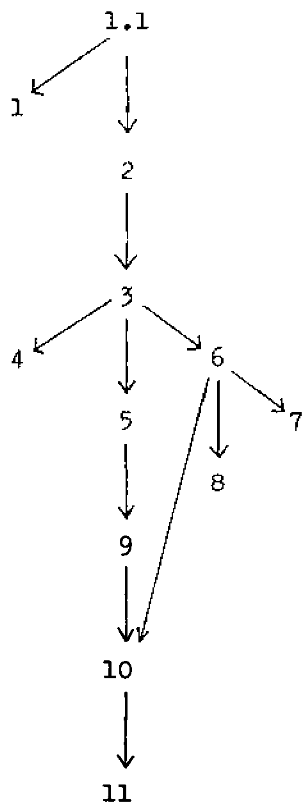

SOME NOTATION AND TERMINOLOGY

Let G be a group.

If S is a subset of a G-group M (p.1), $\langle G\langle S\rangle$ is the G-subgroup generated by S.

We write $\langle 1\langle S\rangle = \langle S\rangle$ = subgroup generated by S.

$Fr_G(M)$ = G-Frattini group of M (§7.1).

$d_G(M)$ = minimum number of G-generators of M (§7.1).

$N_G(A)$ = normalizer of A in G.

$C_G(A)$ = centralizer of A in G.

$|G|$, $|x|$ = order of G, x.

A complete set of representatives of the (right) cosets of A in G is called a (right) transversal of A in G.

G[M = M]G = split extension with kernel M and complement G (§1.1).

Π = product (Cartesian product).

$\coprod$ = coproduct. This is the direct sum in Mod_G, the category of G-modules. It is *, the free product, in the category of groups.

$[a,b] = a^{-1}b^{-1}ab$ = commutator of a by b.

If A, B are subsets of G, $[A,B] = \langle [a,b] \mid a\in A,\ b\in B\rangle$.

If H, K are subgroups of G and $K \triangleleft H$ (normal), H/K is a _factor_ of G. If $[H,G] \leq K$, the factor is called _central_.

A _finite series_ is a family of subgroups (S_i; $0 \leq i \leq m$), where $S_i \triangleleft S_{i+1}$.

If all factors are central, the series is called a _central series_.

If G has a finite central series from 1 to G (i.e., $S_0 = 1$ and $S_m = G$), then G is called _nilpotent_.

If G has a finite series from 1 to G with all factors abelian (cyclic), then G is called _soluble_ (_polycyclic_).

h(G) = Hirsch number of the locally polycyclic G (§8.8).

If $\zeta_0(G) = 1$, $\zeta_1(G)$ = centre of G, and $\zeta_{k+1}(G)$ is the unique subgroup so that $\zeta_{k+1}(G)/\zeta_k(G) = \zeta_1(G/\zeta_k(G))$, then $(\zeta_i(G); i \geq 0)$ is called the _upper central series_ of G.

If G is nilpotent and $\zeta_{c-1}(G) < \zeta_c(G) = G$, then c is the _class_ of G.

If $G_1 = G$, $G_{k+1} = [G_k,G]$, then (G_i; $i \geq 1$) is called the _lower central series_ of G.

If $G^{(0)} = G$, $G' = [G,G]$ ($= G_2$) and $G^{(m+1)} = (G^{(m)})'$, then ($G^{(i)}$; $i \geq 0$) is called the _derived series_ of G.