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Numerical Treatment of Partial Differential Equations

Translated and revised by Martin Stynes

 Springer

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