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L. C. Grove
C. T. Benson

Finite Reflection Groups

Second Edition

With 38 Illustrations

Springer Science+Business Media, LLC

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PREFACE TO THE SECOND EDITION

This edition differs from the original mainly by the addition of a seventh chapter, on the classical invariant theory of finite reflection groups. Most of the changes in the original six chapters are corrections of misprints and minor errors. We are indebted, however, to Klaus Benkert of the RWTH Aachen for pointing out to us Proposition 5.1.5, making possible a neater discussion of the positive definiteness of marked graphs. We have also added an appendix listing the Schoenflies and International notations for crystallographic point groups.

Since many beginning German courses in the United States seem no longer to include an introduction to German script, it may be helpful to some readers if the script letters used in Chapter 7 are introduced here with their Roman counterparts.

German Script	$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{F}$	$\mathfrak{I}$	$\mathfrak{K}$	$\mathfrak{L}$	$\mathfrak{P}$	$\mathfrak{Q}$
Roman	a	b	c	F	I	K	L	P	Q

Our thanks go to David Surowski and Dick Pierce for reading drafts of Chapter 7 and suggesting corrections and improvements, to Helen Grove for typing the new chapter and, belatedly, to Sandra Grove for proofreading the first six.

August 1984 L.C.G. AND C.T.B.

PREFACE TO THE FIRST EDITION

This book began as lecture notes for a course given at the University of Oregon. The course, given for undergraduates and beginning graduate students, follows immediately after a conventional course in linear algebra and serves two chief pedagogical purposes. First, it reinforces the students' newly won knowledge of linear algebra by giving applications of several of the theorems they have learned and by giving geometrical interpretations for some of the notions of linear algebra. Second, some students take the course before or concurrently with abstract algebra, and they are armed in advance with a collection of fairly concrete nontrivial examples of groups.

The first comprehensive treatment of finite reflection groups was given by H. S. M. Coxeter in 1934. In [9] he completely classified the groups and derived several of their properties, using mainly geometrical methods. He later included a discussion of the groups in his book *Regular Polytopes* [10]. Another discussion, somewhat more algebraic in nature, was given by E. Witt in 1941 [37]. An algebraic account of reflection groups was presented by P. Cartier in the Chevalley Seminar reports (see [6]). Another has recently appeared in N. Bourbaki's chapters on Lie groups and Lie algebras [3].

Since the sources cited above do not seem to be easily accessible to most undergraduates, we have attempted to give a discussion of finite reflection groups that is as elementary as possible. We have tried to reach a middle ground between Coxeter and Bourbaki. Our approach is algebraic, but we have retained some of the geometrical flavor of Coxeter's approach.

Chapter 1 introduces some of the terminology and notation used later and indicates prerequisites. Chapter 2 gives a reasonably thorough account of *all* finite subgroups of the orthogonal groups in two and three dimensions. The presentation is somewhat less formal than in succeeding chapters. For instance, the existence of the icosahedron is accepted as an empirical fact, and no formal proof of existence is included. Throughout most of Chapter 2 we do not distinguish between groups that are "geometrically indistinguishable," that is, conjugate in the orthogonal group. Very little of the material in Chapter 2 is actually required for the subsequent chapters, but it serves two important purposes: It aids in the development of geometrical insight, and it serves as a source of illustrative examples.

There is a discussion of fundamental regions in Chapter 3. Chapter 4 provides a correspondence between fundamental reflections and fundamental regions via a discussion of root systems. The actual classification and construction of finite reflection groups takes place in Chapter 5, where we have in part followed the methods of E. Witt and B. L. van der Waerden. Generators and relations for finite reflection groups are discussed in Chapter 6. There are historical remarks and suggestions for further reading in a Postlude.

Since we have written with the student in mind we have included considerable detail and a number of illustrative examples. Exercises are included in every chapter but the first. The results of some of the exercises are used in the body of the text. The list of identifications in Exercise 5.7 was worked out by one of our students, Leslie Wilson.

We wish to thank James Humphreys, Otto Kegel, and Louis Solomon for reading the manuscript and making numerous excellent suggestions. We also derived considerable benefit from Charles Curtis's lectures on root systems and Chevalley groups.

July 1970 C.T.B. AND L.C.G.

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