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Mikhael Gromov

Partial Differential Relations



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Foreword

The classical theory of partial differential equations is rooted in physics, where equations (are assumed to) describe the laws of nature. Law abiding functions, which satisfy such an equation, are very rare in the space of all admissible functions (regardless of a particular topology in a function space).

Moreover, some additional (like initial or boundary) conditions often insure the uniqueness of solutions. The existence of these is usually established with some *apriori estimates* which locate a possible solution in a given function space.

We deal in this book with a completely different class of partial differential equations (and more general relations) which arise in differential geometry rather than in physics. Our equations are, for the most part, undetermined (or, at least, behave like those) and their solutions are rather dense in spaces of functions.

We solve and classify solutions of these equations by means of direct (and not so direct) geometric constructions.

Our exposition is elementary and the proofs of the basic results are selfcontained. However, there is a number of examples and exercises (of variable difficulty), where the treatment of a particular equation requires a certain knowledge of pertinent facts in the surrounding field.

The techniques we employ, though quite general, do not cover all geometrically interesting equations. The border of the unexplored territory is marked by a number of open questions throughout the book.

I am grateful to my friends and colleagues with whom I have discussed various aspects of the subject in the course of years. The book took final shape under unrelenting criticism by Nico Kuiper directed at earlier drafts. I thank Mme V. Houillet for typing the manuscript, Mari Anne Gazdick for rectifying my English and Mme J. Martin for the help with a multitude of last minute corrections.

Bures-sur-Yvette, May 1986

M. Gromov

Contents

Part 1. A Survey of Basic Problems and Results	1
1.1 Solvability and the Homotopy Principle	1
1.1.1 Jets, Relations, Holonomy	1
1.1.2 The Cauchy-Riemann Relation, Oka's Principle and the Theorem of Grauert	4
1.1.3 Differentiable Immersions and the h -Principle of Smale-Hirsch	6
1.1.4 Osculating Spaces and Free Maps	8
1.1.5 Isometric Immersions of Riemannian Manifolds and the Theorems of Nash and Kuiper	10
1.2 Homotopy and Approximation	13
1.2.1 Classification of Solutions by Homotopy and the Parametric h -Principle	13
1.2.2 Density of the h -Principle in the Fine Topologies	18
1.2.3 Functionally Closed Relations	22
1.3 Singularities and Non-singular Maps	26
1.3.1 Singularities as Differential Relations	26
1.3.2 Genericity, Transversality and Thom's Equisingularity Theorem	30
1.4 Localization and Extension of Solutions	35
1.4.1 Local Solutions of Differential Relations	35
1.4.2 The h -Principle for Extensions; Flexibility and Micro-flexibility	39
1.4.3 Ordinary Differential Equations and "Zero-Dimensional" Relations	44
1.4.4 The h -Principle for the Cauchy Extension Problem	46
Part 2. Methods to Prove the h -Principle	48
2.1 Removal of Singularities	48
2.1.1 Immersions and k -Mersions $V \rightarrow \mathbb{R}^q$ for $q > k$	48
2.1.2 Immersions and Submersions $V \rightarrow W$	52
2.1.3 Folded Maps $V^n \rightarrow W^q$ for $q \leq n$	54
2.1.4 Singularities and the Curvature of Smooth Maps	61
2.1.5 Holomorphic Immersions of Stein Manifolds	65

2.2	Continuous Sheaves	74
2.2.1	Flexibility and the h -Principle for Continuous Sheaves	75
2.2.2	Flexibility and Micro-flexibility of Equivariant Sheaves	78
2.2.3	The Proof of the Main Flexibility Theorem	80
2.2.4	Equivariant Microextensions	84
2.2.5	Local Compressibility and the Proof of the Microextension Theorem	87
2.2.6	An Application: Inducing Euclidean Connections	93
2.2.7	Non-flexible Sheaves	98
2.3	Inversion of Differential Operators	114
2.3.1	Linearization and the Linear Inversion	114
2.3.2	Basic Properties of Infinitesimally Invertible Operators	117
2.3.3	The Nash (Newton-Moser) Process	121
2.3.4	Deep Smoothing Operators	123
2.3.5	The Existence and Convergence of Nash's Process	131
2.3.6	The Modified Nash Process and Special Inversions of the Operator \mathcal{D}	139
2.3.7	Infinite Dimensional Representations of the Group $\text{Diff}(V)$	145
2.3.8	Algebraic Solution of Differential Equations	148
2.4	Convex Integration	168
2.4.1	Integrals and Convex Hulls	168
2.4.2	Principal Extensions of Differential Relations	174
2.4.3	Ample Differential Relations	180
2.4.4	Fiber Connected Relations and Directed Immersions	183
2.4.5	Directed Embeddings and the Relative h -Principle	189
2.4.6	Convex Integration of Partial Differential Equations	194
2.4.7	Underdetermined Evolution Equations	195
2.4.8	Triangular Systems of P.D.E.	198
2.4.9	Isometric C^1 -Immersions	201
2.4.10	Isometric Maps with Singularities	207
2.4.11	Equidimensional Isometric Maps	214
2.4.12	The Regularity Problem and Related Questions in the Convex Integration	219
Part 3. Isometric C^∞-Immersions		221
3.1	Isometric Immersions of Riemannian Manifolds	221
3.1.1	Nash's Twist and Approximate Immersions; Isometric Embeddings into \mathbb{R}^q	221
3.1.2	Isometric Immersions $V^n \rightarrow W^q$ for $q \geq (n+2)(n+5)/2$	224
3.1.3	Convex Cones in the Space of Metrics	231
3.1.4	Inducing Forms of Degree $d > 2$	232
3.1.5	Immersions with a Prescribed Curvature	235

3.1.6	Extensions of Isometric Immersions	240
3.1.7	Isometric Immersions $V^n \rightarrow W^q$ for $q \geq (n+2)(n+3)/2$	247
3.1.8	Isometric Cylinders $V^n \times \mathbb{R} \rightarrow W^q$ for $q \geq (n+2)(n+3)/2$	250
3.1.9	Non-free Isometric Maps.	254
3.2	Isometric Immersions in Low Codimension	259
3.2.1	Parabolic Immersions	260
3.2.2	Hyperbolic Immersions	269
3.2.3	Geometric Obstructions to Isometric C^2 -Immersions $V^2 \rightarrow \mathbb{R}^3$	279
3.2.4	Isometric C^∞ -Immersions $V^2 \rightarrow \mathbb{R}^q$ for $3 \leq q \leq 6$	289
3.3	Isometric C^∞-Immersions of Pseudo-Riemannian Manifolds	306
3.3.1	Local Pseudo-Riemannian Immersions	307
3.3.2	Global Immersions	312
3.3.3	Immersion with a Prescribed Curvature and the C^1 -Approximation	316
3.3.4	Isotropic Maps and Non-unique Isometric Immersions	321
3.3.5	Isometric C^∞ -Immersions $V^n \rightarrow W^q$ for $q \geq [n(n+3)/2] + 2$	324
3.4	Symplectic Isometric Immersions	327
3.4.1	Immersion of Exterior Forms	328
3.4.2	Symplectic Immersions and Embeddings	333
3.4.3	Contact Manifolds and Their Immersions	338
3.4.4	Basic Problems in the Symplectic Geometry	340
	References	350
	Author Index	359
	Subject Index	361