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Phillip A. Griffiths

**Exterior Differential  
Systems  
and the Calculus  
of Variations**

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To the memory of my mother

Jeanette Field Griffiths

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## LIST OF COMMONLY USED NOTATIONS

(Note: The references for the undefined terms used below may be found in the index.)

$A^*(X)$	Exterior algebra of smooth differential forms on a manifold $X$
$\{\Sigma\}$	Algebraic ideal in $A^*(X)$ generated by a set $\Sigma$ of forms on $X$
$(I, \omega)$	Exterior differential system with independence condition
$V(I, \omega)$	Set of integral manifolds of $(I, \omega)$
$T_N(V(I, \omega))$	Tangent space to $V(I, \omega)$ at $N$
$(I, \omega; \varphi)$	Variational Problem (cf. Chapter I, Sec. a)
$\Phi: V(I, \omega) \rightarrow \mathbb{R}$	Functional on $V(I, \omega)$
$\delta\Phi: T_N(V(I, \omega)) \rightarrow \mathbb{R}$	Differential of $\Phi$
$V(I, \omega; [A, B])$	Subset of $V(I, \omega)$ given by endpoint conditions
$T_N(V(I, \omega; [A, B]))$	Tangent space to $V(I, \omega; [A, B])$
$\equiv \text{mod } I$	Congruence modulo an ideal $I \subset A^*(X)$
$\equiv$	Congruence modulo the image of $I \wedge I \rightarrow A^*(X)$ (cf. (II.b.4))
<b>PE</b>	Projectivization of a vector space $E$
$\theta_N$	Restriction of $\theta \in A^*(X)$ to a submanifold $N \subset X$
$d\theta = \Theta$	Exterior derivative of a differential form; little $\theta$ is frequently denoted by capital $\Theta$
$F(\cdot)$	Frame manifold
$L_v \varphi$	Lie derivative of a form $\varphi$ along a vector field $v$
$Y$	Momentum space associated to $(I, \omega; \varphi)$
$Q$	Reduced momentum space associated to $(I, \omega; \varphi)$