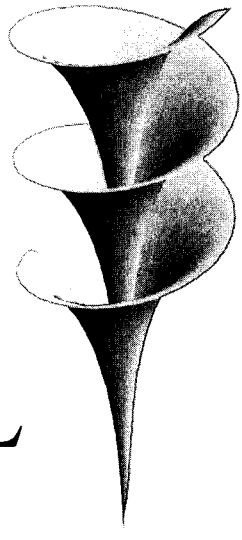
Alfred Gray



**Second Edition**

# MODERN DIFFERENTIAL GEOMETRY of Curves and Surfaces with MATHEMATICA®

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# CONTENTS


**1. Curves in the Plane ..... 1**

1.1 Euclidean Spaces ..... 2

1.2 Curves in $\mathbb{R}^n$ ..... 5

1.3 The Length of a Curve ..... 7

1.4 Vector Fields along Curves ..... 12

1.5 Curvature of Curves in the Plane ..... 14

1.6 Angle Functions Between Plane Curves ..... 17

1.7 The Turning Angle ..... 19

1.8 The Semicubical Parabola ..... 21

1.9 Exercises ..... 22

**2. Studying Curves in the Plane with Mathematica ..... 25**

2.1 Computing Curvature of Curves in the Plane ..... 29

2.2 Computing Lengths of Curves ..... 33

2.3 Filling Curves ..... 34

2.4 Examples of Curves in $\mathbb{R}^2$ ..... 35

2.5 Plotting Piecewise Defined Curves ..... 42

2.6 Exercises ..... 45

*2.7 Animations of Definitions of Curves ..... 47*

***3. Famous Plane Curves ..... 49***

*3.1 Cycloids ..... 50*

*3.2 Lemniscates of Bernoulli ..... 52*

*3.3 Cardioids ..... 54*

*3.4 The Catenary ..... 55*

*3.5 The Cissoid of Diocles ..... 57*

*3.6 The Tractrix ..... 61*

*3.7 Clothoids ..... 64*

*3.8 Pursuit Curves ..... 66*

*3.9 Exercises ..... 70*

*3.10 Animation of a Prolate Cycloid ..... 74*

***4. Alternate Methods for Plotting Plane Curves ..... 75***

*4.1 Implicitly Defined Curves in $\mathbb{R}^2$ ..... 75*

*4.2 Cassinian Ovals ..... 82*

*4.3 Plane Curves in Polar Coordinates ..... 86*

*4.4 Exercises ..... 92*

*4.5 Animation of Fermat's Spiral ..... 96*

***5. New Curves from Old ..... 97***

*5.1 Evolutes ..... 98*

*5.2 Iterated Evolutes ..... 101*

*5.3 The Evolute of a Tractrix is a Catenary ..... 102*

*5.4 Involutes ..... 103*

5.5 Tangent and Normal Lines to Plane Curves ..... 108
5.6 Osculating Circles to Plane Curves ..... 111
5.7 Parallel Curves ..... 115
5.8 Pedal Curves ..... 117
5.9 Exercises ..... 120
5.10 Animation of a Cycloidial Pendulum ..... 126

**6. Determining a Plane Curve from its Curvature ..... 127**

6.1 Euclidean Motions ..... 128
6.2 Curves and Euclidean Motions ..... 134
6.3 Intrinsic Equations for Plane Curves ..... 136
6.4 Drawing Plane Curves with Assigned Curvature ..... 140
6.5 Exercises ..... 146
6.6 Animation of Epicycloids and Hypocycloids ..... 152

**7. Global Properties of Plane Curves ..... 153**

7.1 The Total Signed Curvature of a Plane Curve ..... 154
7.2 The Rotation Index of a Closed Plane Curve ..... 159
7.3 Convex Plane Curves ..... 163
7.4 The Four Vertex Theorem ..... 165
7.5 Curves of Constant Width ..... 168
7.6 Envelopes of Curves ..... 172
7.7 The Support Function of an Oval ..... 173
7.8 Reuleaux Polygons ..... 176
7.9 The Involute Construction of Curves of Constant Width ..... 177
7.10 Exercises ..... 178

**8. Curves in Space ..... 181**

8.1 The Vector Cross Product on $\mathbb{R}^3$ ..... 182

8.2 Curvature and Torsion of Unit-Speed Curves in $\mathbb{R}^3$ ..... 183

8.3 Curvature and Torsion of Arbitrary-Speed Curves in $\mathbb{R}^3$ ..... 189

8.4 Computing Curvature and Torsion with Mathematica ..... 194

8.5 The Helix and its Generalizations ..... 198

8.6 Viviani's Curve ..... 201

8.7 Exercises ..... 202

8.8 Animation of the Frenet Frame of Viviani's Curve ..... 206

**9. Tubes and Knots ..... 207**

9.1 Tubes about Curves ..... 207

9.2 Torus Knots ..... 209

9.3 Exercises ..... 215

**10. Construction of Space Curves ..... 217**

10.1 The Fundamental Theorem of Space Curves ..... 217

10.2 Drawing Space Curves with Assigned Curvature ..... 222

10.3 Contact ..... 225

10.4 Space Curves that Lie on a Sphere ..... 231

10.5 Curves of Constant Slope ..... 234

10.6 Loxodromes on Spheres ..... 238

10.7 Exercises ..... 240

10.8 Animation of Curves on a Sphere ..... 243

**11. Calculus on Euclidean Space ..... 245**

11.1 Tangent Vectors to $\mathbb{R}^n$ ..... 246

11.2 Tangent Vectors as Directional Derivatives ..... 248

11.3 Tangent Maps ..... 250

11.4 Vector Fields on $\mathbb{R}^n$ ..... 255

11.5 Derivatives of Vector Fields on $\mathbb{R}^n$ ..... 258

11.6 Curves Revisited ..... 265

11.7 Exercises ..... 266

**12. Surfaces in Euclidean Space ..... 269**

12.1 Patches in $\mathbb{R}^n$ ..... 269

12.2 Patches in $\mathbb{R}^3$ ..... 277

12.3 The Local Gauss Map ..... 279

12.4 The Definition of a Regular Surface in $\mathbb{R}^n$ ..... 281

12.5 Tangent Vectors to Regular Surfaces in $\mathbb{R}^n$ ..... 286

12.6 Surface Mappings ..... 288

12.7 Level Surfaces in $\mathbb{R}^3$ ..... 291

12.8 Exercises ..... 293

**13. Examples of Surfaces ..... 295**

13.1 The Graph of a Function of Two Variables ..... 296

13.2 The Ellipsoid ..... 301

13.3 The Stereographic Ellipsoid ..... 302

13.4 Tori ..... 304

13.5 The Paraboloid ..... 307

13.6 Sea Shells ..... 308

*13.7 Patches with Singularities ..... 309*

*13.8 Implicit Plots of Surfaces ..... 311*

*13.9 Exercises ..... 312*

***14. Nonorientable Surfaces ..... 317***

*14.1 Orientability of Surfaces ..... 317*

*14.2 Nonorientable Surfaces Described by Identifications ..... 322*

*14.3 The Möbius Strip ..... 325*

*14.4 The Klein Bottle ..... 327*

*14.5 A Different Klein Bottle ..... 329*

*14.6 Realizations of the Real Projective Plane ..... 330*

*14.7 Coloring Surfaces with* Mathematica *..... 335*

*14.8 Exercises ..... 337*

*14.9 Animation of Steiner's Roman Surface ..... 340*

***15. Metrics on Surfaces ..... 341***

*15.1 The Intuitive Idea of Distance on a Surface ..... 341*

*15.2 Isometries and Conformal Maps of Surfaces ..... 346*

*15.3 The Intuitive Idea of Area on a Surface ..... 351*

*15.4 Programs for Computing Metrics and Areas on a Surface ..... 353*

*15.5 Examples of Metrics ..... 354*

*15.6 Exercises ..... 356*

***16. Surfaces in 3-Dimensional Space ..... 359***

*16.1 The Shape Operator ..... 360*

*16.2 Normal Curvature ..... 363*

*16.3 Calculation of the Shape Operator* ..... *367*
*16.4 The Eigenvalues of the Shape Operator* ..... *371*
*16.5 The Gaussian and Mean Curvatures* ..... *373*
*16.6 The Three Fundamental Forms* ..... *380*
*16.7 Examples of Curvature Calculations by Hand* ..... *382*
*16.8 A Global Curvature Theorem* ..... *386*
*16.9 Exercises* ..... *387*

***17. Surfaces in 3-Dimensional Space via* Mathematica ... *391***
*17.1 Programs for Computing the Shape Operator and Curvature* ... *392*
*17.2 Examples of Curvature Calculations with* Mathematica ..... *395*
*17.3 Principal Curvatures via* Mathematica ..... *402*
*17.4 The Gauss Map via* Mathematica ..... *403*
*17.5 The Curvature of Nonparametrically Defined Surfaces* ..... *409*
*17.6 Exercises* ..... *415*

***18. Asymptotic Curves on Surfaces* ..... *417***
*18.1 Asymptotic Curves* ..... *418*
*18.2 Examples of Asymptotic Curves* ..... *422*
*18.3 Using* Mathematica *to Find Asymptotic Curves* ..... *426*
*18.4 Exercises* ..... *429*

***19. Ruled Surfaces* ..... *431***
*19.1 Examples of Ruled Surfaces* ..... *432*
*19.2 Flat Ruled Surfaces* ..... *439*
*19.3 Tangent Developables* ..... *441*

*19.4 Noncylindrical Ruled Surfaces ..... 445*
*19.5 Examples of Striction Curves of Noncylindrical Ruled Surfaces . 449*
*19.6 A Program for Ruled Surfaces ..... 450*
*19.7 Other Examples of Ruled Surfaces ..... 452*
*19.8 Exercises ..... 454*

***20. Surfaces of Revolution ..... 457***
*20.1 Principal Curves ..... 459*
*20.2 The Curvature of a Surface of Revolution ..... 461*
*20.3 Generating a Surface of Revolution with* Mathematica *..... 465*
*20.4 The Catenoid ..... 467*
*20.5 The Hyperboloid of Revolution ..... 470*
*20.6 Surfaces of Revolution of Curves with Specified Curvature ..... 471*
*20.7 Surfaces of Revolution Generated by Data ..... 472*
*20.8 Generalized Helicoids ..... 475*
*20.9 Exercises ..... 478*

***21. Surfaces of Constant Gaussian Curvature ..... 481***
*21.1 The Elliptic Integral of the Second Kind ..... 482*
*21.2 Surfaces of Revolution of Constant Positive Curvature ..... 482*
*21.3 Surfaces of Revolution of Constant Negative Curvature ..... 486*
*21.4 Flat Generalized Helicoids ..... 490*
*21.5 Dini's Surface ..... 493*
*21.6 Kuen's Surface ..... 496*
*21.7 Exercises ..... 497*

**22. Intrinsic Surface Geometry ..... 501**

*22.1 Intrinsic Formulas for the Gaussian Curvature ..... 502*

*22.2 Gauss's Theorema Egregium ..... 507*

*22.3 Christoffel Symbols ..... 509*

*22.4 Geodesic Curvature and Torsion ..... 513*

*22.5 Exercises ..... 519*

**23. Differentiable Manifolds ..... 521**

*23.1 The Definition of Differentiable Manifold ..... 522*

*23.2 Differentiable Functions on Differentiable Manifolds ..... 526*

*23.3 Tangent Vectors on Differentiable Manifolds ..... 532*

*23.4 Induced Maps ..... 540*

*23.5 Vector Fields on Differentiable Manifolds ..... 545*

*23.6 Tensor Fields on Differentiable Manifolds ..... 550*

*23.7 Exercises ..... 554*

**24. Riemannian Manifolds ..... 557**

*24.1 Covariant Derivatives ..... 558*

*24.2 Indefinite Riemannian Metrics ..... 564*

*24.3 The Classical Treatment of Metrics ..... 567*

*24.4 Exercises ..... 572*

**25. Abstract Surfaces ..... 573**

*25.1 Metrics and Christoffel Symbols for Abstract Surfaces ..... 575*

*25.2 Examples of Metrics on Abstract Surfaces ..... 578*

*25.3 Computing Curvature of Metrics on Abstract Surfaces ..... 580*

*25.4 Orientability of an Abstract Surface* ..... *582*

*25.5 Geodesic Curvature for Abstract Surfaces* ..... *583*

*25.6 The Mean Curvature Vector Field* ..... *589*

*25.7 Exercises* ..... *591*

***26. Geodesics on Surfaces ..... 595***

*26.1 The Geodesic Equations* ..... *596*

*26.2 Clairaut Patches* ..... *604*

*26.3 Finding Geodesics Numerically with* Mathematica ..... *613*

*26.4 The Exponential Map and the Gauss Lemma* ..... *617*

*26.5 Length Minimizing Properties of Geodesics* ..... *621*

*26.6 An Abstract Surface as a Metric Space* ..... *623*

*26.7 Exercises* ..... *625*

***27. The Gauss-Bonnet Theorem ..... 627***

*27.1 The Local Gauss-Bonnet Theorem* ..... *628*

*27.2 Topology of Surfaces* ..... *634*

*27.3 The Global Gauss-Bonnet Theorem* ..... *636*

*27.4 Applications of the Gauss-Bonnet Theorem* ..... *638*

*27.5 Exercises* ..... *640*

***28. Principal Curves and Umbilic Points ..... 641***

*28.1 The Differential Equation for the Principal Curves of a Surface* .. *642*

*28.2 Umbilic Points* ..... *645*

*28.3 The Peterson-Mainardi-Codazzi Equations* ..... *649*

*28.4 Hilbert's Lemma and Liebmann's Theorem* ..... *652*

*28.5 The Fundamental Theorem of Surfaces ..... 654*

*28.6 Exercises ..... 659*

***29. Triply Orthogonal Systems of Surfaces ..... 661***

*29.1 Examples of Triply Orthogonal Systems ..... 662*

*29.2 Curvilinear Patches and Dupin's Theorem ..... 665*

*29.3 Elliptic Coordinates ..... 669*

*29.4 Parabolic Coordinates ..... 676*

*29.5 Exercises ..... 678*

***30. Minimal Surfaces ..... 681***

*30.1 Normal Variation ..... 681*

*30.2 Examples of Minimal Surfaces ..... 684*

*30.3 The Gauss Map of a Minimal Surface ..... 694*

*30.4 Exercises ..... 696*

*30.5 Animations of Minimal Surfaces ..... 699*

***31. Minimal Surfaces and Complex Variables ..... 701***

*31.1 Isothermal Coordinates ..... 702*

*31.2 Isometric Deformations of Minimal Surfaces ..... 705*

*31.3 Complex Derivatives ..... 710*

*31.4 Elementary Complex Vector Algebra ..... 714*

*31.5 Minimal Curves ..... 716*

*31.6 Finding Conjugate Minimal Surfaces ..... 721*

*31.7 Enneper's Surface of Degree $n$ ..... 728*

*31.8 Exercises ..... 732*

*31.9 Animations of Minimal Surfaces ..... 734*

**32. Minimal Surfaces via the Weierstrass Representation 735**

*32.1 The Weierstrass Representation ..... 736*

*32.2 Weierstrass Patches via* Mathematica *..... 740*

*32.3 Examples of Weierstrass Patches ..... 741*

*32.4 Minimal Surfaces with One Planar End ..... 744*

*32.5 Costa's Minimal Surface ..... 747*

*32.6 Exercises ..... 760*

**33. Minimal Surfaces via Björling's Formula .....761**

*33.1 Björling's Formula ..... 761*

*33.2 Minimal Surfaces from Plane Curves ..... 764*

*33.3 Examples of Minimal Surfaces Constructed from Plane Curves . 766*

*33.4 Exercises ..... 772*

**34. Construction of Surfaces ..... 773**

*34.1 Parallel Surfaces ..... 773*

*34.2 Parallel Surfaces to a Möbius Strip ..... 777*

*34.3 The Shape Operator of a Parallel Surface ..... 779*

*34.4 A General Construction of a Triply Orthogonal System .....782*

*34.5 Pedal Surfaces ..... 784*

*34.6 Twisted Surfaces ..... 786*

*34.7 Exercises ..... 788*

**35. Canal Surfaces and Cyclides of Dupin ..... 789**

35.1 Surfaces Whose Focal Sets are 2-Dimensional ..... 791

35.2 Canal Surfaces ..... 798

35.3 Cyclides of Dupin via Focal Sets ..... 809

35.4 Exercises ..... 817

**36. Inversions of Curves and Surfaces ..... 819**

36.1 The Definition of Inversion ..... 819

36.2 Inversion of Curves ..... 822

36.3 Inversion of Surfaces ..... 825

36.4 Cyclides of Dupin via Inversions ..... 830

36.5 Liouville's Theorem ..... 832

36.6 Exercises ..... 834

**Appendices ..... 837**

A General Programs ..... 837

B Curves ..... 891

Parametrically Defined Plane Curves ..... 891

Implicitly Defined Plane Curves ..... 916

Polar Defined Plane Curves ..... 923

Parametrically Defined Space Curves ..... 927

C Surfaces ..... 935

Parametrically Defined Surfaces ..... 935

Implicitly Defined Surfaces ..... 969

Drum Plots ..... 974

Minimal Curves ..... 976

Surface Metrics ..... 984

*D Plotting Programs* ..... *987*

*Miscellaneous Plotting Programs* ..... *987*

*Mathematica to Acrospin Programs* ..... *994*

*Mathematica to Geomview Programs* ..... *999*

***Bibliography*** ..... ***1005***

***Index*** ..... ***1021***

***Name Index*** ..... ***1041***

***Miniprogram and* Mathematica *Command Index*** ..... ***1045***