Grundlehren der mathematischen Wissenschaften 236

A Series of Comprehensive Studies in Mathematics

H. Grauert R. Remmert

Theory of Stein Spaces

Translated by Alan Huckleberry

Springer Science+Business Media, LLC

Hans Grauert
Mathematisches Institut
der Universität Göttingen
D-3400 Göttingen
Federal Republic of Germany

Reinhold Remmert
Mathematisches Institut der
Westfälischen Wilhelms-Universität
D-4400 Münster
Federal Republic of Germany

Translator:
Alan Huckleberry
Department of Mathematics
University of Notre Dame
Notre Dame, Indiana 46556
USA

AMS Subject Classifications: 30A46, 32E10, 32J99, 32-02, 32A10, 32A20, 32C15

With 5 Figures

Library of Congress Cataloging in Publication Data
Grauert, Hans, 1930–
Theory of Stein spaces.

(Grundlehren der mathematischen Wissenschaften; 236)
Translation of Theorie der Steinschen Räume.
Includes index.
1. Stein spaces. I. Remmert, Reinhold, joint author. II. Title. III. Series: Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen; 236.
QA331.G68313 515'.73 79-1430

Originally published by Springer Berlin Heidelberg New York in 1979
Softcover reprint of the hardcover 1st edition 1979

9 8 7 6 5 4 3 2 1

ISBN 978-1-4757-4359-3 ISBN 978-1-4757-4357-9 (eBook)
DOI 10.1007/978-1-4757-4357-9

Dedicated to Karl Stein

Contents

Chapter II. Differential Forms and Dolbeault Theory

Chapter III. Theorems A and B for Compact Blocks $\mathbb{C}^m$

Chapter IV. Stein Spaces

Introduction

1. The classical theorem of Mittag-Leffler was generalized to the case of several complex variables by Cousin in 1895. In its one variable version this says that, if one prescribes the principal parts of a meromorphic function on a domain in the complex plane $\mathbb{C}$, then there exists a meromorphic function defined on that domain having exactly those principal parts. Cousin and subsequent authors could only prove the analogous theorem in several variables for certain types of domains (e.g. product domains where each factor is a domain in the complex plane). In fact it turned out that this problem can not be solved on an arbitrary domain in $\mathbb{C}^m$, $m \geq 2$. The best known example for this is a "notched" bicylinder in $\mathbb{C}^2$. This is obtained by removing the set $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| \geq \frac{1}{2}, |z_2| \leq \frac{1}{2}\}$, from the unit bicylinder, $\Delta := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < 1, |z_2| < 1\}$. This domain D has the property that every function holomorphic on D continues to a function holomorphic on the entire bicylinder. Such a phenomenon never occurs in the theory of one complex variable. In fact, given a domain $G \subset \mathbb{C}$, there exist functions holomorphic on G which are singular at *every* boundary point of G. In several complex variables one calls such domains (i.e. domains on which there exist holomorphic functions which are singular at every boundary point) *domains of holomorphy*. H. Cartan observed in 1934 that every domain in $\mathbb{C}^2$ where the above "Cousin problem" is always solvable is necessarily a domain of holomorphy. A proof of this was communicated by Behnke and Stein in 1937. Meanwhile it was conjectured that Cousin's theorem should hold on any domain of holomorphy. This was in fact proved by Oka in 1937: For every prescription of principal parts on a domain of holomorphy $D \subset \mathbb{C}^m$, there exists a meromorphic function on D having exactly those principal parts. In the same year, via the example of $\mathbb{C}^3 \setminus \{0\}$, H. Cartan showed that it is possible for the Cousin theorem to be valid on domains which are not domains of holomorphy.

As the theory of functions of several complex variables developed, it was often the case that, in order to have a chance of carrying over important one variable results, it was necessary to restrict to domains of holomorphy. This was particularly true with respect to the analog of the Weierstrass product theorem. Formulated as a question, it is as follows: Given a domain D in $\mathbb{C}^m$, can one prescribe the zeros (counting multiplicity) of a holomorphic function on D? It was soon realized that in some cases it is impossible to find even a continuous function which does the job. Conditions for the existence of a continuous solution of this problem, the so-called "second Cousin problem," were discussed by K. Stein in 1941. In fact he gave a sufficient condition which could actually be checked in particular examples. Nowadays this is stated in terms of the vanishing of the Chern class of the prescribed zero set. Stein, however, stated this in a dual and more intuitively geometric way. His condition is as follows: The "intersection number" of the zero surface (counting multiplicity) with any 2-cycle in D should always be zero.

It was similarly necessary to restrict to domains of holomorphy in order to prove the appropriate generalizations of the facts that, on a domain in $\mathbb{C}$, every meromorphic function is the ratio of (globally defined) analytic functions and, if the domain is simply connected, holomorphic functions can be uniformly approximated by polynomials (i.e. the Runge approximation theorem). Poincaré first posed the question about meromorphic functions of several variables being quotients of globally defined relatively prime holomorphic functions. He in fact answered this positively in certain interesting cases (e.g. for $\mathbb{C}^m$ itself).

2. It is not at all straightforward to generalize the notion of a Mittag-Leffler distribution (i.e. prescriptions of principal parts) to the several variable case. The main difficulty is that the set on which the desired function is to have poles is no longer discrete. In fact, in the case of domains in $\mathbb{C}^m$, $m \geq 2$, this set is a $(2m-2)$-dimensional real (possibly singular) surface. Thus one can no longer just prescribe points and pieces of Laurent series. This can be circumvented as follows: If G is a domain in $\mathbb{C}^m$ and $\mathfrak{U} = \{U_i\}$, $i \in I$, is an open covering of G, then the family $\{U_i, h_i\}$ is called an *additive Cousin distribution* on G, whenever each h_i is a meromorphic function on U_i, and on $U_{i_0 i_1} := U_{i_0} \cap U_{i_1}$ the difference $h_{i_0} - h_{i_1}$ is holomorphic for all choices of i_0 and i_1. In the case of $m = 1$, this means that h_{i_0} and h_{i_1} have the same principal parts. Thus one obtains a Mittag-Leffler distribution from the Cousin distribution. A meromorphic function h is said to have the Cousin distribution for its principal parts if $h - h_i$ is holomorphic on U_i for all i.

Different Cousin distributions can, on the same covering, define the same distribution of principal parts. This difficulty is overcome by introducing an equivalence relation. For this let $x \in G$. Let U be an open neighborhood of x in G and suppose that h is meromorphic on U. Then the pair (U, h) is called a *locally meromorphic function* at x. Two such pairs (U_1, h_1) and (U_2, h_2) are called equivalent if there exists a neighborhood V of x with $V \subset U_1 \cap U_2$ and $h_1 - h_2$ holomorphic on V. Each equivalence class is called a *germ of a principal part*. The set of all germs of principal parts at x is denoted by $\mathscr{H}_x$. We define $\mathscr{H} := \bigcup_{x \in X} \mathscr{H}_x$ and denote by $\pi: \mathscr{H} \to G$ the map which associates to every germ its base point $x \in G$. If $U \subset G$ is open and h is meromorphic on U then, for every $x \in U$, one has the associated principal part of h at x, $\bar{h}_x \in \mathscr{H}_x$. Consequently there exists a map $s_h: U \to \mathscr{H}$, $x \mapsto \bar{h}_x$, such that $\pi \cdot s_h = id$. It is easy to check that sets of the form $s_h(U)$, where U is any open set in G and h is any meromorphic function on U, form a basis for a topology on $\mathscr{H}$. Further, in this topology, $\pi: \mathscr{H} \to G$ is seen to be continuous and a local homeomorphism. In such a situation one calls $\mathscr{H}$ a sheaf over G. The fibers of π should be thought of as stalks with the open sets looking like transversal surfaces given by the maps s_h. The map $s_h: U \to \mathscr{H}$ is called a local section over U. Every Cousin distribution $\{U_i, h_i\}$ defines a *global* continuous map (section) $s: G \to \mathscr{H}$ with $\pi \cdot s = id$. This is locally defined by $s \mid U_i := s_{h_i}$. The condition that, for all i and j, $h_i - h_j$ is holomorphic on $U_i \cap U_j$ is equivalent to the fact that s is well-defined. Two Cousin distributions have the same principal parts if and only if they correspond to the same section in D over G. A meromorphic function h is a "solution" of the Cousin distribution s (i.e. has exactly the same principal parts as were prescribed) exactly when $s_h = s$.

It is clear from the above that the sheaf theoretic language is the ideal medium for the statement of the generalization of the Mittag-Leffler problem to the several variable situation. Of course for domains in $\mathbb{C}^n$ Oka had solved this without explicit use of sheaves. But even in this case the language of sheaves isolated the real problems and made the seemingly complicated techniques of Oka more transparent. This was also true in the case of the second Cousin problem, the Poincaré problem, etc. Furthermore this language was ideal for formulating new problems and for paving the road toward possible obstructions to their solutions. Theorems about sheaves themselves later gave rise to numerous interesting applications.

3. The germs of holomorphic functions form a sheaf which is usually denoted by $\mathcal{O}$. It has already been pointed out that the zero sets of analytic functions are important even in the study of the Cousin problems. Thus it should be expected that analytic sets, which are just sets of simultaneous zeros of finitely many holomorphic functions on domains in the various $\mathbb{C}^m$, would play an important role in the early development of the theory. In fact the totality of germs of holomorphic functions which vanish on a particular analytic set form a subsheaf of $\mathcal{O}$ which frequently comes into play in present day complex analysis. In 1950 Oka himself used the idea of distributions of ideals in rings of local holomorphic functions (idéaux de domaines indéterminés). This notion, which at the time of its conception seemed difficult and mysterious, just corresponds to the simple idea of a sheaf of ideals.

The use of germs and the idea of sheaves go back to the work of J. Leray. Sheaves have been systematically applied in the theory of functions of several complex variables ever since 1950/51. The idea of *coherence* is very important for many considerations in several complex variables. Roughly speaking, a sheaf of $\mathcal{O}$-modules is coherent if it is locally free except possibly on some small set where it is still finitely generated with the ring of relations again being finitely generated. Even in the early going it was necessary to prove the coherence of many sheaves. This was often quite difficult, because there were really no techniques around and most work had to be done from scratch. The most important coherence theorems originated with H. Cartan and K. Oka. After the foundations had been laid, coherent sheaves quickly enriched the theory of domains of holomorphy with new important results. In the meantime, in his memorable work "Analytische Functionen mehrerer komplexer Veränderlichen zu vorgegebenen Periodizitätsmoduln und das zweite Cousinsche Problem," Math. Ann. **123**(1951), 201–222, K. Stein had discovered complex manifolds which have basic (elementary) properties similar to domains of holomorphy. A domain $G \subset \mathbb{C}$ is indeed a domain of holomorphy if and only if it is a *Stein manifold*. The main point is that many theorems about coherent sheaves on domains of holomorphy can as well be proved for Stein manifolds. Cartan and Serre recognized that the language of sheaf cohomology, which had been developed only shortly before, is particularly suitable for the formulation of the main results: For every coherent sheaf $\mathscr{S}$ over a Stein manifold X, the following two theorems hold:

Theorem A. The $\mathcal{O}(X)$-module of global sections $\mathscr{S}(X)$ generates every stalk $\mathscr{S}_x$ as an $\mathcal{O}_x$-module for all $x \in X$.

Theorem B. $H^q(X, \mathscr{S}) = 0$ for all $q \geq 1$.

These famous theorems, which were first proved in the Seminaire Cartan 1951/52, contain, among many others, the results pertaining to the Cousin problems.

4. Following the original definition, a *paracompact* complex manifold is called a *Stein manifold* if the following three axioms are satisfied:

Separation Axiom: *Given two distinct points $x_1, x_2 \in X$, there exists a function f holomorphic on X such that $f(x_1) \neq f(x_2)$.*

Local Coordinates Axiom: *If $x_0 \in X$ then there exists a neighborhood U of x_0 and functions $f_1, \ldots, f_m$ which are holomorphic on X such that the restrictions $z_i := f_i \mid U$, $i = 1, \ldots, m$, give local coordinates on U.*

Holomorphic Convexity Axiom: *If $\{x_i\}$ is a sequence which "goes to ∞ in X" (i.e. the set $\{x_i\}$ is discrete) then there exists a function f holomorphic on X which is unbounded on $\{x_i\}$*: $\sup |f(x_i)| = \infty$.

It is clear that a domain in $\mathbb{C}^m$ is a Stein manifold if and only if it is holomorphically convex. However if one wants to study non-schlicht domains over $\mathbb{C}^m$ (i.e. ramified covers of domains in $\mathbb{C}^m$), then it is not apriori clear that two points lying over the same base point can be separated by global holomorphic functions. Likewise it is not obvious that neighborhoods of ramification points have local coordinates which are restrictions of global holomorphic functions. If one allows points which are not locally uniformizable (i.e. points where there is a genuine singularity and the "domain" is not even a manifold, as is the case at the point $(0, 0, 0) \in V := \{(x, y, z) \in \mathbb{C}^3 \mid x^2 = yz\}$, which is spread over the (y, z)-plane by projection) then the above definition is meaningless, because we assumed that X is a manifold. However, even in the non-locally uniformizable situation above, the following significant weakening of the separation and local coordinate axioms still holds:

Weak Separation Axiom: *For every point $x_0 \in X$ there exist functions $f_1, \dots, f_n \in \mathcal{O}(X)$ so that x_0 is an isolated point in $\{x \in X \mid f_1(x) = \cdots = f_n(x) = 0\}$.*

Among other things, this allows the consideration of spaces with singularities. Due to the maximum principle, this weak separation implies that *every compact analytic subspace of X is finite.*

It turns out that, without losing the main results, the convexity axiom can also be somewhat weakened:

Weak Convexity Axiom: *Let K be a compact set in X and W an open neighborhood of K in X. Then $\hat{K} \cap W$ is compact, where $\hat{K}$ denotes the holomorphic hull of K in X:*

$$\hat{K} := \{x \in X \mid |f(x)| \leq \sup_{y \in K} |f(y)|, \quad \text{for all} \quad f \in \mathcal{O}(X)\}.$$

One way of strengthening the axiom immediately above is to require that $\hat{K}$ be compact in X. If one does this and further considers only the case where X is a manifold, then, without the use of deep techniques, one can show that the strengthened axiom is equivalent to the holomorphic convexity axiom (see Theorems IV.2.4 and IV.2.12).

For the purposes of this book, a Stein space is a *paracompact* (*not necessarily reduced*) *complex space for which Theorem B is valid.* It is proved that this condition is equivalent to the validity of Theorem A, and is also equivalent to the above weakened axioms. In particular it follows that if X is a manifold, the weakened axioms imply Stein's original axioms.

We will always assume that a complex space has *countable topology* and is thus *paracompact.* With a bit of work one can show that any irreducible complex space which satisfies the weak separation axiom is eo ipso paracompact (see 16, 24).

5. We conclude our introductory remarks with a short description of the contents of this book. We begin with two brief preliminary chapters (Chapters A and B) where we assemble the important information from sheaf theory and the related cohomology theories. The idea of coherence is explained in these chapters. A reader who is really interested in coherence proofs, can find such in our book, "Coherent Analytic Sheaves," which is presently in preparation. Complex spaces are introduced as special $\mathbb{C}$-algebraized spaces. Further we develop cohomology from the point of view of alternating (Čech) cochains as well as via flabby resolutions. Proofs which are easily accessible in the literature (e.g. [SCV], [TF], or [TAG]) are in general not carried out.

In Chapter I a short direct proof of the coherence theorem for finite holomorphic maps is given. It is based primarily on the Weierstrass division theorem and Hensel's lemma for convergent power series.

The Dolbeault cohomology theory is presented in Chapter II. As a consequence we obtain Theorem B for the structure sheaf $\mathcal{O}$ over a compact euclidean block (i.e. an m-fold product of rectangles), K, in $\mathbb{C}^m$. In other words, for $q \geq 1$, $H^q(K, \mathcal{O}) = 0$. It should be noted that, although we want to introduce Dolbeault cohomology in any case, this result follows directly and with less difficulty via the Čech cohomology.

Chapter III contains the proofs for Theorems A and B for coherent sheaves over euclidean blocks $K \subset \mathbb{C}^m$. One of the key ingredients for the proofs is the fact that, for every coherent sheaf $\mathscr{S}$, the cohomology groups, $H^q(K, \mathscr{S})$, vanish for all q large enough. The deciding factor in proving Theorem A is the "Heftungslemma" of Cartan. This is proved quite easily if while solving the Cousin problem, one simultaneously estimates the attaching functions.

In Chapter IV Theorems A and B are proved for an arbitrary Stein space, X. A summary of the proof is the following: First it is shown that X is exhausted by analytic blocks. (An analytic block is a compact set in X which can be mapped by a finite, proper, holomorphic map into an euclidean block in some $\mathbb{C}^m$.) The coherence theorem for finite maps along with the results in Chapter III yield the desired theorem free of charge. In order to obtain such theorems in the limit (i.e. for spaces exhausted by analytic blocks), an approximation technique, which is a generalization of the usual Runge idea, is needed.

Applications and illustrations of the main theorems, as well as examples of Stein manifolds, are given in Chapter V. The canonical Fréchet topology on the space of global sections $\mathscr{S}(X)$ of a coherent analytic sheaf is described in Section 4. By means of the normalization theorem, which we do not prove in this book, we give a simple proof for the fact that, for a reduced complex space X, the canonical Fréchet topology on $H^0(X, \mathscr{O})$ is the topology of compact convergence.

Chapter VI is devoted to proving that, for a coherent analytic sheaf $\mathscr{S}$ on a compact complex space X, $H^q(X, \mathscr{S})$, $q \geq 0$, are finite dimensional $\mathbb{C}$-vector space (Théorème de finitude of Cartan and Serre). In this proof we work with the Hilbert space of square-integrable holomorphic functions and make use of the orthonormal basis which was introduced by S. Bergman. The classical Schwarz lemma plays an important role, replacing the lemma of L. Schwartz on linear compact maps between Fréchet spaces.

In Chapter VII we attempt to entertain the reader with a presentation of the theory of compact Riemann surfaces which results from, among other considerations, the finiteness theorem of Chapter V. The celebrated Riemann-Roch and Serre duality theorems are proved. The flow of the proof is more or less like that in Serre [35], except that, in the analytic case, a real argument for $H^1(X, \mathscr{M}) = 0$ is needed. This is done in a simple way using an idea of R. Kiehl. The book closes with a proof of the Grothendieck theorem on the splitting of vector bundles over $\mathbb{C}P_1$.

The reader should be advised that, while the English version is not a word for word translation of Theorie der Steinschen Räume, there are no significant changes in the mathematics. There are a number of strategies for reading this book, depending on the experience and viewpoint of the reader. Those who are not currently working the field might first browse through the chapter on applications (Chapter V).

It gives us great pleasure to be able to dedicate this book to Karl Stein, who initiated the theory as well as collaborated in its development. Various preliminary versions of our texts were already in existence in the middle 60's. We would like to thank W. Barth for his help at that time.

It is our pleasure to express sincere thanks to Professor Dr. Alan Huckleberry from the University of Notre Dame, South Bend, Indiana, for translating this book into English.

Göttingen, Münster/Westf. H. Grauert R. Remmert