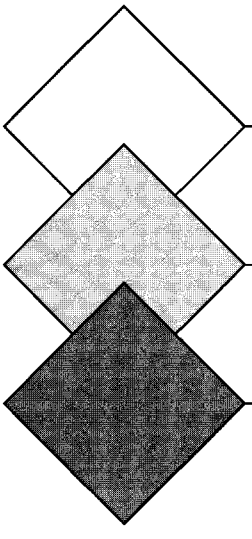
George Grätzer



General Lattice Theory

Second edition

New appendices with B.A. Davey, R. Freese, B. Ganter,
M. Greferath, P. Jipsen, H.A. Priestley, H. Rose, E.T. Schmidt,
S.E. Schmidt, F. Wehrung, R. Wille

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