

Georges Gras

# Class Field Theory

From Theory to Practice



Springer

*Georges Gras*  
University of Franche-Comté  
Faculty of Sciences  
Laboratory of Mathematics and CNRS  
16, route de Gray  
25030 Besançon Cedex, France  
e-mail: gras@math.univ-fcomte.fr or g.mn.gras@wanadoo.fr

*Translator of the original French manuscript*  
Henri Cohen  
University of Bordeaux I  
Mathematics and Computer Sciences  
351, Cours de la Libération  
33405 Talence Cedex, France

Library of Congress Cataloging-in-Publication Data applied for

Bibliographic information published by Die Deutsche Bibliothek  
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;  
detailed bibliographic data is available in the Internet at <<http://dnb.ddb.de>>.

---

Mathematics Subject Classification (2000): 11RXX, 12R37, 11R29, 11R70, 11R34, 11S31, 11Y40

---

ISSN 1439-7382  
ISBN 978-3-642-07908-5 ISBN 978-3-662-11323-3 (eBook)  
DOI 10.1007/978-3-662-11323-3

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag Berlin Heidelberg GmbH.  
Violations are liable for prosecution under the German Copyright Law.

<http://www.springer.de>

© Springer-Verlag Berlin Heidelberg 2003  
Originally published by Springer-Verlag Berlin Heidelberg New York in 2003  
Softcover reprint of the hardcover 1st edition 2003

The use of general descriptive names, registered names, trademarks etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: *Erich Kirchner*, Heidelberg

Typesetting: by the author

Printed on acid-free paper

SPIN 10883751

41/3142ck-543210

# Table of Contents

<b>Preface</b> .....	vii
<b>Introduction to Global Class Field Theory</b> .....	1
<b>I. Basic Tools and Notations</b> .....	7
§1 Places of $K$ .....	9
§2 Embeddings of a Number Field in its Completions .....	12
§3 Number and Ideal Groups .....	21
a) The Local Case: The Group $K_v^\times$ .....	21
b) The Global Case: Numbers, Ideals, and Units .....	23
§4 Idèle Groups — Generalized Class Groups .....	27
a) Idèle Groups — Topology .....	28
b) Generalized Class Groups — Rank Formulas .....	37
§5 Reduced Idèles — Topological Aspects .....	45
a) The Fundamental Exact Sequence .....	45
b) Topological Lemmas .....	49
c) Characters of Profinite Groups .....	53
§6 Kummer Extensions .....	54
a) Algebraic Kummer Theory .....	54
b) Arithmetic Aspects of Kummer Theory .....	59
<b>II. Reciprocity Maps — Existence Theorems</b> .....	65
§1 The Local Reciprocity Map — Local Class Field Theory .....	65
a) Decomposition of Places: Local and Global Cases .....	66
b) Local Class Field Theory Correspondence .....	74
c) Local Conductors and Norm Groups .....	80
d) Infinite Local Class Field Theory .....	86
§2 Idèle Groups in an Extension $L/K$ .....	91
a) Canonical Injection of $C_K$ in $C_L$ .....	91
b) Relations Between Local and Global Norms .....	92
c) Galois Structure of $J_L$ : Semi-Local Theory .....	94
d) Local Norm Groups — The Non-Galois Case .....	98
§3 Global Class Field Theory: Idelic Version .....	104
a) Global Reciprocity Map — The Product Formula — Global Class Field Theory Correspondence .....	104

- b) Global Class Field Theory in  $\overline{K}^{\text{ab}}/K$  ..... 121
- §4 Global Class Field Theory: Class Group Version ..... 125
  - a) Global Norm Conductor — Properties ..... 125
  - b) Artin’s Reciprocity Map — Reciprocity Law — Global  
Computation of Hasse Symbols — Decomposition Law 130
- §5 Ray Class Fields — Hilbert Class Fields ..... 143
  - a) Elementary Properties — Decomposition Law ..... 144
  - b) Rank Formulas — The Reflection Theorem ..... 152
  - c) Class Field Theory Over  $\mathbb{Q}$  ..... 161
  - d) Congruence Groups ..... 164
  - e) Norm Action on Generalized Class Groups ..... 164
  - f) The Principal Ideal Theorem — Hilbert Towers ..... 168
- §6 The Hasse Principle — For Norms — For Powers ..... 176
- §7 Symbols Over Number Fields — Hilbert and Regular Kernels 195

**III. Abelian Extensions with Restricted Ramification — Abelian**

- Closure** ..... 221
- §1 Generalities on  $H_T^S/H^S$  and its Subextensions ..... 221
  - a) Description of  $\text{Gal}(K_{(\mathfrak{m})}^S/H^S)$  ..... 221
  - b) The Case of  $p$ -Extensions ..... 226
  - c) The Structure of  $\text{Gal}(H_T^S/H^S)$  —  $p$ -Adic Ranks ..... 233
- §2 Computation of  $\mathcal{A}_T^S := \text{Gal}(H_T^S(p)/K)$  and  $\mathcal{T}_T^S := \text{tor}_{\mathbb{Z}_p}(\mathcal{A}_T^S)$  . 240
  - a)  $\mathbb{Z}_p$ -Free-Extensions — Logarithms ..... 240
  - b)  $\mathcal{A}_T^S$  as an Infinitesimal Ray Class Group ..... 243
  - c) Computation of  $\mathcal{T}_T^S$  ..... 250
  - d) Class Field Theory Correspondence in  $H_T^{\text{res}}(p)/K$  ..... 256
- §3 Compositum of the  $S$ -Split  $\mathbb{Z}_p$ -Extensions — The  $p$ -Adic  
Conjecture ..... 258
  - a)  $p$ -Adic Ranks: The Leopoldt–Jaulent–Roy Conjecture . 258
  - b) The Galois Case ..... 264
  - c) The Monogeneous Case ..... 268
- §4 Structure Theorems for the Abelian Closure of  $K$  ..... 274
  - a) Deployment of  $\text{Gal}(\overline{K}_{(p)}^{\text{ab}}/H_p^{\text{ord}}(p))$  ..... 275
  - b) Triviality Criterion for  $\mathcal{T}_T^{\text{ord}}$ : When is  $\mathcal{G}_T^{\text{ord}}$  Pro- $p$ -Free? 282
  - c) The Schmidt–Chevalley Theorem — Inertia Groups in  
 $\overline{K}_{(p)}^{\text{ab}}/K$  ..... 287
  - d) Galois Diagram for  $\overline{K}_{(p)}^{\text{ab}}/K$  — Structure of the Con-  
nected Component  $D_0$  — The Fundamental Equality:  
 $\overline{K}_v^{\text{ab}} = (\overline{K}^{\text{ab}})_v$  ..... 291
  - e) Decomposition Law of Wild Places in  $\overline{K}_{(p)}^{\text{ab}}/H_p^{\text{ord}}(p)$  . 300
  - f) The Strong  $p$ -Adic Conjecture — Other  $p$ -Adic Aspects 305
  - g) Structural Properties of  $\overline{G}^{\text{ab}}$  — Divisibility of the Con-  
nected Component — Cyclic Embedding Criterion . . . . 323

h) The Grunwald–Wang Theorem — Weak Deployment  
 Theorem for Decomposition Groups . . . . . 330

§5 Explicit Computations in Incomplete  $p$ -Ramification . . . . . 342

§6 Initial Radical of the  $\mathbb{Z}_p$ -Extensions . . . . . 348

§7 The Logarithmic Class Group . . . . . 354

**IV. Invariant Class Groups in  $p$ -Ramification — Genus Theory** 361

§1 Reduction to the Case of  $p$ -Ramification . . . . . 362

§2 Injectivity of the Transfer Map  $\mathcal{A}_K^{\text{ord}} \rightarrow \mathcal{A}_L^{\text{ord}}$  . . . . . 363

§3 Determination of  $(\mathcal{A}_L^{\text{ord}})^G$  and  $(\mathcal{T}_L^{\text{ord}})^G$  —  $p$ -Rational Fields . . 365

a) Invariant Classes Formulas . . . . . 366

b)  $p$ -Primitive Ramification —  $p$ -Rationality . . . . . 371

§4 Genus Theory with Ramification and Decomposition . . . . . 375

a) Computation of the Number of Genera — Examples . . 377

b) The Genus Exact Sequence . . . . . 390

c) Central Classes — Knot Groups . . . . . 398

**V. Cyclic Extensions with Prescribed Ramification** . . . . . 407

§1 Study of an Example . . . . . 408

§2 Construction of a Governing Field . . . . . 410

a) Solution to the Cyclic Case of Degree  $p$  . . . . . 412

b) Minimal Ramification Sets . . . . . 421

c) Approach to the Cyclic Case of Degree  $p^e$  . . . . . 423

d) Solution to the Weak Form . . . . . 432

§3 Conclusion and Perspectives . . . . . 434

**Appendix: Arithmetical Interpretation of  $H^2(\mathcal{G}_T^S, \mathbb{Z}/p^e\mathbb{Z})$**  . 441

§1 A General Approach by Class Field Theory . . . . . 442

a) Study of  $\text{Ker}(H^2(G, \mathbb{Z}/p^e\mathbb{Z}) \xrightarrow{\text{Inf}} H^2(\mathcal{G}_T^S, \mathbb{Z}/p^e\mathbb{Z}))$  . . . 443

b) Study of  $H^2(G, \mathbb{Z}/p^e\mathbb{Z})$  — The Schur Multiplier . . . . 444

c) A Class Field Theory Formula for  $|\text{Inf}(H^2(G, \mathbb{Z}/p^e\mathbb{Z}))|$  . 449

§2 Complete  $p$ -Ramification Without Finite Decomposition . . . . 450

§3 The General Case — Infinitesimal Knot Groups . . . . . 453

a) Infinitesimal Computations . . . . . 454

b) Infinitesimal Knot Groups — The Number of Relations — A Generalization of Šafarevič’s Results . . . . 456

c) Finite Generalized  $p$ -Class Fields Towers . . . . . 460

d) A Lower Bound for  $\text{rk}_p(H^2(\mathcal{G}_T^S, \mathbb{Z}/p\mathbb{Z}))$  — Conclusion . 463

**Bibliography** . . . . . 467

**Index of Notations** . . . . . 481

**General Index** . . . . . 487