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Essays in Commutative Harmonic Analysis

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To my wife, Jill Wescott Graham

To my father and mother,
Oscar M. McGehee and Louise Blanche Carruth McGehee

Preface

This book considers various spaces and algebras made up of functions, measures, and other objects—situated always on one or another locally compact abelian group, and studied in the light of the Fourier transform. The emphasis is on the objects themselves, and on the structure-in-detail of the spaces and algebras.

A mathematician needs to know only a little about Fourier analysis on the commutative groups, and then may go many ways within the large subject of harmonic analysis—into the beautiful theory of Lie group representations, for example. But this book represents the tendency to linger on the line, and the other abelian groups, and to keep asking questions about the structures thereupon. That tendency, pursued since the early days of analysis, has defined a field of study that can boast of some impressive results, and in which there still remain unanswered questions of compelling interest.

We were influenced early in our careers by the mathematicians Jean-Pierre Kahane, Yitzhak Katznelson, Paul Malliavin, Yves Meyer, Joseph Taylor, and Nicholas Varopoulos. They are among the many who have made the field a productive meeting ground of probabilistic methods, number theory, diophantine approximation, and functional analysis. Since the academic year 1967–1968, when we were visitors in Paris and Orsay, the field has continued to see interesting developments. Let us name a few. Sam Drury and Nicholas Varopoulos solved the union problem for Helson sets, by proving a remarkable theorem (2.1.3) which has surely not seen its last use. Gavin Brown and William Moran and others fleshed out the framework that Joseph Taylor had provided for the study of convolution algebras, and Thomas Körner's construction techniques made child's play of problems once thought intractable.

The book is for those who work in commutative harmonic analysis, for those who wish to do so, and for those who merely want to look into it. In the areas that we have chosen to treat, we have tried to make more accessible than before not only the results for their own sakes, but also the techniques, the points of view, and the sources of intuition by which the subject lives.

We have had repeatedly to choose whether to present material in the abstract setting of an arbitrary locally compact abelian group G, or on, say, the circle group T. As often as not, restricting the discussion to a concrete setting makes the essential ideas more vivid, and one loses nothing but technical clutter. But sometimes one must concede the greater usefulness and aesthetic appeal of a general treatment. So we have made sometimes the one choice, and sometimes the other. But let us emphasize that the subject is truly the union, not the intersection, of the studies on the various abelian groups.

The order of the chapters does not have the usual significance, even though we did choose it with care. One reviewer suggests that 12 and 11 should appear between 4 and 5. In any event, whenever the material of one chapter depends on some part of another one, the reader is provided with a specific reference. Therefore one who is not discouraged by the Prerequisites, and who is familiar with our Symbols, Conventions, and Terminology, may begin reading at any one of the chapters.

We thank our home departments, at Northwestern and Louisiana State, for their support over the years. We thank also the several other mathematics departments where one or both of us have visited and found pleasant conditions for work: in Paris, Jerusalem, Urbana, Eugene, and Honolulu.

We thank the many colleagues and friends who have given us encouragement and help. In particular, for their extensive and critical attention to drafts of various parts of the book, we thank Aharon Atzmon, John Fournier, Yitzhak Katznelson, Thomas Ramsey, and George Shapiro. Especially do we thank Sadahiro Saeki, who read over half the book with care and made many valuable suggestions.

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Prerequisites

The areas in which it is most important for the reader to have both knowledge and facility are as follows.

1. Basic functional analysis, as in Dunford and Schwartz [1, Chapter II and Sections V.1–V.6] or Rudin [3].
2. The theory of measure and integration, as in Royden [1, Parts 1 and 3].
3. Commutative Banach algebra theory, as in Rudin [3, Chapter 11].
4. Fourier analysis on the line and the circle, as in Katznelson [1, Chapters I, IV, and VI; also Sections II.1 and V.1].
5. Fourier analysis on locally compact abelian groups, as in Rudin [1, Chapters 1 and 2]. In particular, we shall use the structure theorem: every locally compact abelian group G has an open subgroup of the form $R^n \times H$, where $n \geq 0$ and H is compact. For another treatment of that theorem, see Hewitt and Ross [1, Section 24].

In addition, the reader will find it helpful to have sampled the theory of exceptional subsets ("thin sets") of groups, as for example in Lindahl and Poulsen [1, Chapter 1] and Kahane [1, Chapters III and IV].

Some of the elementary material is treated in the Appendix. For example, the results of Section 2.6 in Rudin [1] are given a different treatment in A.5.

Besides the works that we have recommended here, there are of course other excellent sources from which to acquire the same background knowledge.

There are isolated places in the book where we use other, more advanced and specialized material, and at such points we give specific references.

Symbols, Conventions, and Terminology

Before beginning any of the chapters, the reader should study this list of symbols and terms that are used most frequently. Each item is attended by a brief definition, and perhaps also a remark or two about relevant conventions and basic facts. Some of the definitions make use of others on the list. The order is alphabetical, with the Greek entries grouped all together after the Latin ones; except that we single out several items to explain at the outset.

The symbol G stands for an arbitrary locally compact abelian group, except when some other meaning is specified. The same is true for the symbol Γ. When G and Γ appear in the same context, each denotes the dual group of the other; and then for $x \in G$ and $\gamma \in \Gamma$, the value of γ at x is denoted by $\langle x, \gamma \rangle$. Thus if Γ is considered as an additive group, $\langle x, \gamma_1 + \gamma_2 \rangle = \langle x, \gamma_1 \rangle \cdot \langle x, \gamma_2 \rangle$. If f is an element of a Banach space and S an element of the dual space, then too, $\langle f, S \rangle$ means the value of S at f.

The symbol E nearly always stands for a closed subset of Γ. Whenever $X = X(\Gamma)$ is a Banach algebra of functions on Γ (such as A, AP, B, B_0, or M_p), the symbol $X(E)$ (or $X(E, \Gamma)$) stands for the Banach algebra of restrictions to E of functions in X with norm

$$\|f\|_{X(E)} = \inf\{\|g\|_X : g = f \quad \text{on} \quad E\}.$$

Equivalently, $X(E)$ may be defined as the quotient algebra X/I, where I is the ideal $\{f \in X : f^{-1}(0) \supseteq E\}$. But when X is a space of distributions on G (such as M, M_1, M_c, M_d, PF, or PM), then the symbol $X(E)$ stands for the subspace of X consisting of the elements with support contained in E.

$A(\Gamma)$	—the Fourier representation of the convolution algebra $L^1(G)$; that is, the Banach algebra of Fourier transforms $\hat{f}$ of elements f of $L^1(G)$. The operators are the usual pointwise ones, and the norm, denoted by $\|\hat{f}\|_{A(\Gamma)}$ or $\|\hat{f}\|_A$, is defined to equal the $L^1(G)$-norm of f. Note the natural norm-decreasing inclusion: $A(\Gamma) \subseteq C_o(\Gamma)$.
$AP(\Gamma)$	—the algebra of almost periodic functions on Γ, with pointwise operations and the supremum norm. It is realizable as $C(b\Gamma)$.

$B(\Gamma)$ —the Fourier representation of the convolution algebra $M(G)$; that is, the Banach algebra of Fourier transforms $\hat{\mu}$ of measures $\mu \in M(G)$. The elements of $B(\Gamma)$ are also called the Fourier–Stieltjes transforms on Γ. The operations are the usual pointwise ones, and the norm, denoted by $\|\hat{\mu}\|_{B(\Gamma)}$ or $\|\hat{\mu}\|_B$, is the total variation of μ. Note the natural isometric and isomorphic inclusions: $L^1(G) \subseteq M(G)$, $A(\Gamma) \subseteq B(\Gamma)$.

$B_o(\Gamma)$ $= B(\Gamma) \cap C_o(\Gamma)$.

BV —the space of functions of bounded variation

$b\Gamma$ —the Bohr compactification of Γ; equivalently, the dual group of G_d.

$\mathbb{C}$ —the complex number system.

$C(X)$ —where X is a topological space: the Banach algebra of bounded continuous complex-valued functions on X, with the usual pointwise operations and the supremum norm.

$C_c(X)$ — $\{f \in C(X)$: the support of f is compact$\}$.

$C_o(X)$ —the subalgebra of $C(X)$ consisting of the functions that vanish at infinity.

countable —in one-to-one correspondence with some subset of the positive integers.

D —the Cantor group; that is, the product group $\prod_{j=1}^{\infty} G_j$, where each G_j is the two-element group.

$\# E$ *or* Card E —the cardinality of the set E.

E-polynomial, E-function, E-measure Let E be a subset of Γ. An E-polynomial, E-function, or E-measure is a trigonometric polynomial, a function, or a measure, respectively, whose Fourier transform vanishes on the complement of E.

$\hat{f}$ —(1) the Fourier transform of f, where f is a function, bounded Borel measure, or distribution defined on (say) G. Thus if $f \in L^1(G)$,

$$\hat{f}(\gamma) = \int_G \langle x, -\gamma \rangle dm_G(x) \qquad \text{for } \gamma \in \Gamma.$$

More generally, for $\mu \in M(G)$,

$$\hat{\mu}(\gamma) = \int_G \langle x, -\gamma \rangle d\mu(x) \qquad \text{for } \gamma \in \Gamma.$$

The Fourier transform provides isometric isomorphisms $L^1(G) \triangleq A(\Gamma)$, $M(G) \triangleq B(\Gamma)$, since $(\mu_1 * \mu_2)^\wedge = \hat{\mu}_1 \hat{\mu}_2$.

	—(2) the Gelfand transform of an element f of a Banach algebra.
F_σ	—the class of all sets that are countable unions of closed sets.
G_δ	—the class of all sets that are countable intersections of open sets.
$\hat{G}$	—the dual group of G.
G_d	—G, but with its topology replaced by the discrete topology.
G^n	—the product group $\prod_{j=1}^n G_j$, where each G_j is G.
$Gp\ H$	—where $H \subseteq G$: the group generated algebraically by H.
Helson set	See $\alpha(E)$ below.
Hermitian	—$\mu \in M(G)$ is Hermitian if $\tilde{\mu} = \mu$.
$h(I)$	—where I is an ideal in a Banach algebra of functions on a set X: the *hull* of I, $h(I) = \{x \in X: f(0) = 0$ for all $f \in I\}$.
$I(E)$, $I_o(E)$, $J(E)$	When $E \subseteq \Gamma$ and $A = A(\Gamma)$ is the algebra under discussion, these symbols denote respectively the largest ideal whose hull is E: $$I(E) = \{f \in A: f^{-1}(0) \supseteq E\},$$ the smallest closed ideal whose hull is E, and the smallest ideal whose hull is E: $J(E) = \{f \in A: f^{-1}(0)$ is a neighborhood of $E\}$. Note that $I_o(E)$ is the closure in A of $J(E)$.
independent	A set $E \subseteq G$ is independent if whenever $x_1, \ldots, x_n \in E$ and $u_1, \ldots, u_n \in Z$, and $\sum_{j=1}^n u_j x_j = 0$, then (1) $u_j x_j = 0$ for $1 \le j \le n$. But when $G = T$, we replace (1) in the definition by (2) $u_j = 0$ for $1 \le j \le n$.
Kronecker set	A set $E \subseteq G$ is a Kronecker set if for every $f \in C(E)$ and every $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that $\|f(x) - \langle x, \gamma\rangle\| < \varepsilon$ for all $x \in E$. A Kronecker set is evidently a Helson set.
K_p-set	A set $E \subseteq G = \prod_{j=1}^\infty G_j$, where each G_j is T_p, is a K_p-set if for every continuous $f: E \to T_p$ there exists

$\gamma \in \Gamma$ such that $f(x) = \langle x, \gamma \rangle$ for all $x \in E$. A K_p-set is evidently a Helson set.

$L^1(G)$ —the convolution algebra of Haar-integrable complex-valued functions (or rather, equivalence classes thereof) on G. Convolution is given by:

$$f * g(x) = \int_G f(x - y)g(y)dm_G(y).$$

The norm, under which $L^1(G)$ is a Banach algebra, is given by:

$$\|f\|_1 = \|f\|_{L^1(G)} = \int_G |f(x)|dm_G(x).$$

$L^p(G)$ $(1 \leq p < \infty)$ —the Banach space of equivalence classes of measurable functions f on G such that $|f|^p$ is integrable, with norm $\|f\|_p = \|f\|_{L^p(G)} = \int_G |f(x)|^p \, dm_G(x)$.

$L^\infty(G)$ —the Banach space of equivalence classes of essentially bounded measurable functions f on G, with norm

$$\|f\|_\infty = \|f\|_{L^\infty(G)} = \inf\{c : |f(x)| \leq c \quad \text{l.a.e.-}m_G\}.$$

$L^p_E(G)$ $= \{f \in L^p(G) : \hat{f}(\gamma) = 0 \text{ for } \gamma \notin E\}$.

m_G —a Haar measure on G, normalized so that $m_G(\{0\}) = 1$ if G is discrete and infinite; or so that $m_G(G) = 1$ if G is compact. We often write dx for $dm_G(x)$, $d\gamma$ for $dm_\Gamma(\gamma)$, and so forth. As for the real line, $dm_R(x)$ is alternatively Lebesgue measure dx, or $dx/2\pi$; thus for $f \in L^1(R)$, $\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-iyx}\, dx$, and if $\hat{f} \in L^1(R)$, $f(x) = (1/2\pi)\int_{-\infty}^{\infty} \hat{f}(y)e^{iyx}\, dy$.

$M(G)$ —the convolution algebra of bounded complex-valued Borel measures on G. Convolution is given by: $\mu * \nu(E) = \int_G \mu(E - x)d\nu(x)$ for every Borel set E. The norm, under which $M(G)$ is a Banach algebra, is given by

$$\|\mu\|_M = \|\mu\|_{M(G)} = \int_G |d\mu(x)|.$$

$M(G)$ is the Banach space dual of $C_o(G)$, with the pairing given by:

$$\langle f, \mu \rangle = \int f(x)\overline{d\mu(x)}.$$

$M(E)$	—where $E \subseteq G$: the subspace of $M(G)$ consisting of the measures with support contained in E.
$M_1(E)$	$= \{\mu \in M(E)$: supp μ is a finite set$\}$.
$M_c(E)$	$= \{\mu \in M(E)$: μ is continuous, that is, $\mu(\{x\}) = 0$ for each $x \in E\}$.
$M_d(E)$	$= \{\mu \in M(E)$: μ is discrete$\} \simeq l^1(E)$.
$M_o(E)$	$= \{\mu \in M(E)$: $\hat{\mu} \in C_o(\Gamma)\}$.
M-set	A set E is an M-set if $PF(E) \neq \{0\}$.
M_o-set	A set E is an M_o-set if $PF(E) \cap M(E) \neq \{0\}$.
$N(E)$	—the annihilator of $I(E)$ in $PM(\Gamma)$; equivalently, the Banach space dual of the quotient algebra $A(E) = A(G)/I(E)$.
Parseval relation	—an identity of a certain kind, of which the following are examples. (1) For $f \in A(G)$ and $\mu \in M(G)$, $\int_G f(x)\overline{d\mu(x)} = \int_\Gamma \hat{f}(\gamma)\overline{\hat{\mu}(\gamma)}d\gamma$. (2) For f, $g \in L^2(G)$, $\int_G f(x)g(x)dx = \int_\Gamma \hat{f}(\gamma)\hat{g}(\gamma)d\gamma$.
Pisot number	—an algebraic integer $\theta > 1$ with conjugates $\theta, x_1, \ldots, x_{n-1}$ such that $\lvert x_j \rvert < 1$ for each j.
portion	If U is an open interval on the line or circle, and if $U \cap E$ is closed and nonempty, then $U \cap E$ is called a portion of E.
$PF(\Gamma)$	—the subspace of $PM(\Gamma)$ consisting of those pseudomeasures S such that $\hat{S} \in C_o(G)$. Its elements are called pseudofunctions. Its dual space is $B(\Gamma)$, which of course equals $A(\Gamma)$ when G is discrete.
$PM(\Gamma)$	—the Banach space dual of $A(\Gamma)$. If $S \in PM(\Gamma)$, S is called a pseudomeasure, and there exists $\hat{S} \in L^\infty(G)$ such that for every $\hat{f} \in A(\Gamma)$, $$\langle \hat{f}, S \rangle = \int_G f(x)\overline{\hat{S}(x)}dx.$$ Conversely, every element $S \in L^\infty(G)$ gives rise to a pseudomeasure, and $\lVert S \rVert_{PM} = \lVert S \rVert_{PM(\Gamma)} = \lVert \hat{S} \rVert_{L^\infty(\Gamma)}$. Note the natural norm-decreasing inclusion: $A(\Gamma) \subseteq C_o(\Gamma)$ and its adjoint: $M(\Gamma) \subseteq PM(\Gamma)$. As usual with a Banach space and its pre-dual, $PM(\Gamma)$ is a module over $A(\Gamma)$; for $S \in PM$ and $f \in A$, we define $fS \in PM$ by: $\langle g, fS \rangle = \langle gf, S \rangle$ for $g \in A$. Note that $\lVert fS \rVert_{PM} \leq \lVert f \rVert_A \lVert S \rVert_{PM}$; and that $(fS)^\wedge = \hat{f} * \hat{S}$.
R	—the real number system.
R^n	—n-dimensional Euclidean space.

Rad I	—the radical of an ideal I in a Banach algebra B; that is, the intersection of the maximal modular ideals of B that contain I. An example when $B = M(G)$: Rad $L^1(G) = \{\mu \in M(G)\colon \hat{\mu}(\psi) = 0$ for all $\psi \in \Delta\backslash\Gamma\}$.
Sidon set	See $\alpha(E)$ below.
supp S	—the support of the distribution S.
T	—the circle group, realized additively as R mod 2π, or multiplicatively as $\{z \in C\colon \|z\| = 1\}$.
T_p	—the subgroup $\{z \in T\colon z^p = 1\}$ (where p is a positive integer).
weak* topology	—the topology $\sigma(X, X^*)$, the weakest topology on the dual space X^* of a Banach space X with respect to which the mapping $x^* \to \langle x, x^*\rangle$ is continuous for every $x \in X$.
Z	—the integer group.
Z_p	—Z mod p (where p is a positive integer).
$\alpha(E)$	—the Helson constant of a closed set $E \subseteq G$, called also the Sidon constant when G is discrete. The set E is a *Helson set* (called a Sidon set when G is discrete) if it is a set of interpolation for the algebra $A(G)$—that is, if $A(E) = C_o(E)$. For arbitrary E, the inclusion map: $A(E) \subseteq C_o(E)$ is one-to-one and norm-decreasing, and $A(E)$ is dense in $C_o(E)$, so that E is a Helson set if and only if the quantity $$\alpha(E) = \sup\left\{\frac{\\|f\\|_{A(E)}}{\\|f\\|_{C_o(E)}} : f \in A(E), f \neq 0\right\}$$ is finite. Evidently $1 \leq \alpha(E) \leq \infty$, and by duality $$\alpha(E) = \sup\left\{\frac{\\|\mu\\|_M}{\\|\mu\\|_{PM}} : \mu \in M(E), \mu \neq 0\right\}.$$ When G is discrete, E is a Helson set (a Sidon set) if and only if $B(E) = l^\infty(E)$.
δ_x or $\delta(x)$	—the measure μ such that $\mu(E) = \begin{cases} 1 \text{ if } x \in E, \\ 0 \text{ if } x \in E. \end{cases}$
ΔB	—the maximal ideal space of the commutative Banach algebra B.
∂B	—the Šilov boundary of ΔB, where B is a commutative Banach algebra; that is, the smallest closed set $S \subseteq \Delta B$ such that for every $f \in B$, $\sup_{h \in \Delta B} \|\hat{f}(h)\|$ is attained at some $h \in S$.

$\tilde{\mu}$	—the conjugate of the measure $\mu \in M(G)$, defined by the condition that $\tilde{\mu}(E) = \overline{\mu(-E)}$ for every Borel set $E \subseteq G$. Note that $\hat{\tilde{\mu}} = \bar{\hat{\mu}}$ on Γ.
$\mu\|_E$	—the restriction to the set E of the measure μ; that is, the measure ν such that $\nu(F) = \mu(E \cap F)$.
μ_c	—the continuous part of the measure μ.
μ_d	—the discrete (or atomic) part of μ.
μ_s	—the singular part of the measure μ.
$\mu \perp \nu$	—means that μ and ν are mutually singular.
$\mu \ll \nu$	—means that μ is absolutely continuous with respect to ν.
$\mu \approx \nu$	—means that $\mu \ll \nu$ and $\nu \ll \mu$.
ΣB	—the set of symmetric maximal ideals in the commutative Banach algebra B, that is, B has an involution $f \mapsto \tilde{f}$ and $\psi \in \Sigma B$ if and only if $\tilde{f}^{\wedge}(\psi) = \hat{f}(\psi)^{-}$ for all $f \in B$. In the case of $M(G)$, it is clear that $\Gamma \subseteq \Sigma M(G)$.
χ_E	—the indicator function (sometimes called the characteristic function) of the set E: $\chi_E(x) = 1$ if $x \in E$, 0 otherwise.