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Stable Mappings and Their Singularities

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PREFACE

This book aims to present to first and second year graduate students a beautiful and relatively accessible field of mathematics—the theory of singularities of stable differentiable mappings.

The study of stable singularities is based on the now classical theories of Hassler Whitney, who determined the generic singularities (or lack of them) for mappings of $R^n \to R^m$ ($m \geq 2n - 1$) and $R^2 \to R^2$, and Marston Morse, who studied these singularities for $R^n \to R$. It was René Thom who noticed (in the late '50's) that all of these results could be incorporated into one theory. The 1960 Bonn notes of Thom and Harold Levine (reprinted in [42]) gave the first general exposition of this theory. However, these notes preceded the work of Bernard Malgrange [23] on what is now known as the Malgrange Preparation Theorem—which allows the relatively easy computation of normal forms of stable singularities as well as the proof of the main theorem in the subject—and the definitive work of John Mather. More recently, two survey articles have appeared, by Arnold [4] and Wall [53], which have done much to codify the new material; still there is no totally accessible description of this subject for the beginning student. We hope that these notes will partially fill this gap. In writing this manuscript, we have repeatedly cribbed from the sources mentioned above—in particular, the Thom-Levine notes and the six basic papers by Mather. This is one of those cases where the hackneyed phrase "if it were not for the efforts of . . . , this work would not have been possible" applies without qualification.

A few words about our approach to this material: We have avoided (although our students may not always have believed us) doing proofs in the greatest generality possible. For example, we assume in many places that certain manifolds are compact and that, in general, manifolds have no boundaries, in order to reduce the technical details. Also, we have tried to give an abundance of low-dimensional examples, particularly in the later chapters. For those topics that we do cover, we have attempted to "fill in all the details," realizing, as our personal experiences have shown, that this phrase has a different interpretation from author to author, from chapter to chapter, and—as we strongly suspect—from authors to readers. Finally, we are aware that there are blocks of material which have been included for completeness' sake and which only a diehard perfectionist would slog through —especially on the first reading although probably on the last as well. Conversely, there are sections which we consider to be right at the "heart of the matter." These considerations have led us to include a Reader's Guide to the various sections.

Chapter I: This is elementary manifold theory. The more sophisticated reader will have seen most of this material already but is advised to glance through it in order to become familiar with the notational conventions used elsewhere in the book. For the reader who has had some manifold theory before, Chapter I can be used as a source of standard facts which he may have forgotten.

Chapter II: The main results on stability proved in the later chapters depend on two deep theorems from analysis: Sard's theorem and the Malgrange preparation theorem. This chapter deals with Sard's theorem in its various forms. In §1 is proved the classical Sard's theorem. Sections 2–4 give a reformulation of it which is particularly convenient for applications to differentiable maps: the Thom transversality theorem. These sections are essential for what follows, but there are technical details that the reader is well-advised to skip on the first reading. We suggest that the reader absorb the notion of k-jets in §2, look over the first part of §3 (through Proposition 3.5) but assume, without going through the proofs, the material in the last half of this section. (The results in the second half of §3 would be easier to prove if the domain X were a compact manifold. Unfortunately, even if we were only to work with compact domains, the stability problem leads us to consider certain noncompact domains like $X \times X - \Delta X$.) In §4, the reader should probably skip the details of the proof of the multijet transversality theorem (Theorem 4.13). It is here that the difficulties with $X \times X - \Delta X$ make their first appearance.

Sections 5 and 6 include typical applications of the transversality theorem. The tubular neighborhood theorem, §7, is a technical result inserted here because it is easy to deduce from the Whitney embedding theorem in §5.

Chapter III: We recommend this chapter be read carefully, as it contains in embryo the main ideas of the stability theory. The first section gives an incorrect but heuristically useful "proof" of the Mather stability theorem: the equivalence of stability and infinitesimal stability. (The theorem is actually proved in Chapter V.) For motivational reasons we discuss some facts about infinite dimensional manifolds. These facts are used nowhere in the subsequent chapters, so the reader should not be disturbed that they are only sketchily developed. In the remaining three sections, we give all the elementary examples of stable mappings. The proofs depend on the material in Chapter II and the yet to be proved Mather criterion for stability.

Chapter IV gives the second main result from analysis needed for the stability theory: the Malgrange preparation theorem. Like Chapter II, this chapter is a little technical. We have provided a way for the reader to get through it without getting bogged down in details: in the first section, we discuss the classical Weierstrass preparation theorem—the holomorphic version of the Malgrange theorem. The proof given is fairly easy to understand, and has the virtue that the adaptation of it to a proof of the Malgrange preparation theorem requires only one additional fact, namely, the Nirenberg extension lemma (Proposition 2.4). The proof of this lemma can probably be skipped by the reader on a first reading as it is hard and technical.

In the third section, the form of the preparation theorem we will be using in subsequent chapters is given. The reader should take some pains to understand it (particularly if his background in algebra is a little shaky, as it is couched in the language of rings and modules).

Chapter V contains the proof of Mather's fundamental theorem on stability. The chapter is divided into two halves; §§1–4 contain the proof that infinitesimal stability implies stability and §§5 and 6 give the converse. In the process of proving the equivalence between these two forms of stability we prove their equivalence with other types of stability as well. For the reader who is confused by the maze of implications we provide in §7 a short summary of our line of argument.

It should be noted that in these arguments we assume the domain X is compact and without boundary. These assumptions could be weakened but at the expense of making the proof more complicated. One pleasant feature of the proof given here is that it avoids Banach manifolds and the global Mather division theorem.

Chapters VI and VII provide two classification schemes for stable singularities. The one discussed in Chapter VI is due to Thom [46] and Boardman [6]. The second scheme, due to Mather and presented in the last chapter, is based on the "local ring" of a map. One of the main results of these two chapters is a complete classification of all equidimensional stable maps and their singularities in dimensions ≤ 4. (See VII, §6.) The reader should be warned that the derivation of the "normal forms" for some stable singularities (VII, §§4 and 5) tend to be tedious and repetitive.

Finally, the *Appendix* contains, for completeness, a proof of all the facts about Lie groups needed for the proofs of Theorems in Chapters V and VI.

This book is intended for first and second year graduate students who have limited—or no—experience dealing with manifolds. We have assumed throughout that the reader has a reasonable background in undergraduate linear algebra, advanced calculus, point set topology, and algebra, and some knowledge of the theory of functions of one complex variable and ordinary differential equations. Our implementation of this assumption—i.e., the decisions on which details to include in the text and which to omit—varied according to which undergraduate courses we happened to be teaching, the time of day, the tides, and possibly the economy. On the other hand, we are reasonably confident that this type of background will be sufficient for someone to read through the volume. Of course, we realize that a healthy dose of that cure-all called "mathematical sophistication" and a previous exposure to the general theory of manifolds would do wonders in helping the reader through the preliminaries and into the more interesting material of the later chapters.

Finally, we note that we have made no attempt to create an encyclopedia of known facts about stable mappings and their singularities, but rather to present what we consider to be basic to understanding the volumes of material that have been produced on the subject by many authors in the past few years. For the reader who is interested in more advanced material, we recommend perusing the volumes of the "Proceedings of Liverpool Singularities" [42, 43], Thom's basic philosophical work, "Stabilité Structurelle et Morphogenèse" [47], Tougeron's work, "Ideaux de Fonctions Differentiables" [50], Mather's forthcoming book, and the articles referred to above.

There were many people who were involved in one way or another with the writing of this book. The person to whom we are most indebted is John Mather, whose papers [26–31] contain almost all the fundamental results of stability theory, and with whom we were fortunately able to consult frequently. We are also indebted to Harold Levine for having introduced us to Mather's work, and, for support and inspiration, to Shlomo Sternberg, Dave Schaeffer, Rob Kirby, and John Guckenheimer. For help with the editing of the manuscript we are grateful to Fred Kochman and Jim Damon. For help with some of the figures we thank Molly Scheffe. Finally, our thanks to Marni Elci, Phyllis Ruby, and Kathy Ramos for typing the manuscript and, in particular, to Marni for helping to correct our execrable prose.

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TABLE OF CONTENTS

Stable Mappings and Their Singularities