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Robert Goldblatt

Lectures on the Hyperreals

An Introduction to Nonstandard Analysis

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Preface

There are good reasons to believe that nonstandard analysis, in some version or other, will be the analysis of the future.

KURT GÖDEL

This book is a compilation and development of lecture notes written for a course on nonstandard analysis that I have now taught several times. Students taking the course have typically received previous introductions to standard real analysis and abstract algebra, but few have studied formal logic. Most of the notes have been used several times in class and revised in the light of that experience. The earlier chapters could be used as the basis of a course at the upper undergraduate level, but the work as a whole, including the later applications, may be more suited to a beginning graduate course.

This preface describes my motivations and objectives in writing the book. For the most part, these remarks are addressed to the potential instructor.

Mathematical understanding develops by a mysterious interplay between intuitive insight and symbolic manipulation. Nonstandard analysis requires an enhanced sensitivity to the particular symbolic form that is used to express our intuitions, and so the subject poses some unique and challenging pedagogical issues. The most fundamental of these is how to turn the *transfer principle* into a working tool of mathematical practice. I have found it unproductive to try to give a proof of this principle by introducing the formal Tarskian semantics for first-order languages and working through the proof of Łoś's theorem. That has the effect of making the subject seem more difficult and can create an artifical barrier to understanding. But the practical use of transfer is more readily explained informally, and typically involves statements that are no more complicated than the "epsilon-delta" statements used in standard analysis. My approach then has been to *illustrate* transfer by many examples, with demonstrations of why those examples work, leading eventually to a situation in which its formulation as a general principle appears quite credible.

There is an obvious analogy with standard laws of thought, such as induction. It would be an unwise teacher who attempted to introduce this to the novice by deriving the principle of induction as a theorem from the axioms of set theory. Of course one attempts to *describe* induction, and *explain* how it is applied. Eventually after practice with examples the student gets used to using it. So too with transfer.

It is sensible to use this approach in many areas of mathematics, for instance beginning a course on standard analysis with a description of the real number system $\mathbb{R}$ as a complete ordered field. The student already has well-developed intuitions about real numbers, and the axioms serve to summarise the essential information needed to proceed. It is rare these days to find a text that begins by explicitly constructing $\mathbb{R}$ out of the rationals via Dedekind cuts or Cauchy sequences, before embarking on the theory of limits, convergence, continuity, etc.

On the other hand, it is not so clear that such a methodology is adequate for the introduction of the hyperreal field $^*\mathbb{R}$ itself. In view of the controversial history of infinitesimals, and the student's lack of familiarity with them, there is a plausibility problem about simply introducing $^*\mathbb{R}$ axiomatically as an ordered field that extends $\mathbb{R}$, contains infinitesimals, and has various other properties. I hope that such a descriptive approach will eventually become the norm, but here I have opted to use the foundational, or constructive, method of presenting an ultrapower construction of the ordered field structure of $^*\mathbb{R}$, and of enlargements of elementary sets, relations, and functions on $\mathbb{R}$, leading to a development of the calculus, analysis, and topology of functions of a single variable. At that point (Part III) the exposition departs from some others by making an early introduction of the notions of internal, external, and hyperfinite subsets of $^*\mathbb{R}$, and internal functions from $^*\mathbb{R}$ to $^*\mathbb{R}$, along with the notions of overflow, underflow, and saturation. It is natural and helpful to develop these important and radically new ideas in this simpler context, rather than waiting to apply them to the more complex objects produced by constructions based on superstructures.

As to the use of superstructures themselves, again I have taken a slightly different tack and followed (in Part IV) a more axiomatic path by positing the existence of a *universe* $\mathbb{U}$ containing all the entities (sets, tuples, relations, functions, sets of sets of functions, etc., etc.) that might be needed in pursuing a particular piece of mathematical analysis. $\mathbb{U}$ is described by set-theoretic closure properties (pairs, unions, powersets, transitive closures). The role of the superstructure construction then becomes the foundational one of showing that universes exist. From the point of view of mathematical practice, enlargements of superstructures seem somewhat artificial (a "gruesome formalism", according to one author), and the approach taken here is intended to make it clearer as to what exactly is the ontology that we need in order to apply nonstandard methods. Looking to the future, if (one would like to say *when*) nonstandard analysis becomes as widely recognised as its standard "shadow", so that a descriptive approach without any need for ultrapowers is more amenable, then the kind of axiomatic account developed here on the basis of universes would, I believe, provide an effective and accessible style of exposition of the subject.

What does nonstandard analysis offer to our understanding of mathematics? In writing these notes I have tried to convey that the answer includes the following five features.

(1) *New definitions of familiar concepts, often simpler and more intuitively natural*

Examples to be found here include the definitions of convergence, boundedness, and Cauchy-ness of sequences; continuity, uniform continuity, and differentiability of functions; topological notions of interior, closure, and limit points; and compactness.

(2) *New and insightful (often simpler) proofs of familiar theorems*

In addition to many theorems of basic analysis about convergence and limits of sequences and functions, intermediate and extreme values and fixed points of continuous functions, critical points and inverses of differentiable functions, the Bolzano–Weierstrass and Heine–Borel theorems, the topology of sets of reals, etc., we will see nonstandard proofs of Ramsey's theorem, the Stone representation theorem for Boolean algebras, and the Hahn–Banach extension theorem on linear functionals.

(3) *New and insightful constructions of familiar objects*

For instance, we will obtain integrals as hyperfinite sums; the reals $\mathbb{R}$ themselves as a quotient of the hyperrationals $^*\mathbb{Q}$; other completions, including the p-adic numbers and standard power series rings as quotients of nonstandard objects; and Lebesgue measure on $\mathbb{R}$ by a nonstandard counting process with infinitesimal weights.

(4) *New objects of mathematical interest*

Here we will exhibit new kinds of number (limited, unlimited, infinitesimal, appreciable); internal and external sets and functions; shadows; halos; hyperfinite sets; nonstandard hulls; and Loeb measures.

(5) *Powerful new properties and principles of reasoning*

These include transfer; internal versions of induction, the least number principle and Dedekind completeness; overflow, underflow, and other principles of permanence; Robinson's sequential lemma; saturation; internal set definition; concurrence; enlargement; hyperfinite approximation; and comprehensiveness.

In short, nonstandard analysis provides us with an enlarged view of the mathematical landscape. It represents yet another stage in the emergence of new number systems, which is a significant theme in mathematical history. Its rich conceptual framework will be built on to reveal new systems and new understandings, so its development will itself influence the course of that history.

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