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Calculus of Variations I

The Lagrangian Formalism

With 73 Figures



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Contents of Calculus of Variations I

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