## Contents

Preface		xi
Chapter	1. A Brief Classical Introduction	1
§1.1.	Quadratic Forms as Polynomials	1
§1.2.	Representation and Equivalence; Matrix Connections;	4
Errorei	Discriminants	4
Exerc		7
$\S{1.3.}$	A Brief Historical Sketch, and Some References to the	-
	Literature	7
Chapter	2. Quadratic Spaces and Lattices	13
$\S{2.1.}$	Fundamental Definitions	13
$\S{2.2.}$	Orthogonal Splitting; Examples of Isometry and Non-isometry	16
Exerc	ises	20
$\S{2.3.}$	Representation, Splitting, and Isotropy; Invariants $u(F)$ and	
	s(F)	21
$\S{2.4.}$	The Orthogonal Group of a Space	26
$\S{2.5.}$	Witt's Cancellation Theorem and Its Consequences	29
$\S{2.6.}$	Witt's Chain Equivalence Theorem	34
$\S{2.7.}$	Tensor Products of Quadratic Spaces; the Witt ring of a field	35
Exerc	ises	39
$\S2.8.$	Quadratic Spaces over Finite Fields	40
$\S{2.9.}$	Hermitian Spaces	44
Exerc	ises	49
	-	vii
		V 11

Chapter	3. Valuations, Local Fields, and <i>p</i> -adic Numbers	51
$\S{3.1.}$	Introduction to Valuations	51
$\S{3.2.}$	Equivalence of Valuations; Prime Spots on a Field	54
Exerc	ises	58
$\S{3.3.}$	Completions, $\mathbb{Q}_p$ , Residue Class Fields	59
$\S{3.4.}$	Discrete Valuations	63
$\S{3.5.}$	The Canonical Power Series Representation	64
$\S{3.6.}$	Hensel's Lemma, the Local Square Theorem, and Local Fields	69
$\S{3.7.}$	The Legendre Symbol; Recognizing Squares in $\mathbb{Q}_p$	74
Exerc	ises	76
Chapter	4. Quadratic Spaces over $\mathbb{Q}_p$	81
$\S4.1.$	The Hilbert Symbol	81
$\S4.2.$	The Hasse Symbol (and an Alternative)	86
$\S4.3.$	Classification of Quadratic $\mathbb{Q}_p$ -Spaces	87
§4.4.	Hermitian Spaces over Quadratic Extensions of $\mathbb{Q}_p$	92
Exerc	ises	94
Chapter 5. Quadratic Spaces over $\mathbb{Q}$ 97		
$\S{5.1.}$	The Product Formula and Hilbert's Reciprocity Law	97
$\S{5.2.}$	Extension of the Scalar Field	98
$\S{5.3.}$	Local to Global: The Hasse–Minkowski Theorem	99
$\S{5.4.}$	The Bruck–Ryser Theorem on Finite Projective Planes	105
$\S{5.5.}$	Sums of Integer Squares (First Version)	109
Exerc	ises	111
Chapter	6. Lattices over Principal Ideal Domains	113
$\S6.1.$	Lattice Basics	114
$\S6.2.$	Valuations and Fractional Ideals	116
$\S 6.3.$	Invariant factors	118
$\S6.4.$	Lattices on Quadratic Spaces	122
$\S6.5.$	Orthogonal Splitting and Triple Diagonalization	124
$\S6.6.$	The Dual of a Lattice	128
Exerc	ises	130
$\S6.7.$	Modular Lattices	133
$\S6.8.$	Maximal Lattices	136
$\S 6.9.$	Unimodular Lattices and Pythagorean Triples	138

§6.10. Remarks on Lattices over More General Rings	141
Exercises	142
Chapter 7. Initial Integral Results	145
$\S7.1.$ The Minimum of a Lattice; Definite Binary Z-Lattices	146
§7.2. Hermite's Bound on min L, with a Supplement for $k[x]$ -La	
§7.3. Djokovič's Reduction of $k[x]$ -Lattices; Harder's Theorem	
§7.4. Finiteness of Class Numbers (The Anisotropic Case)	156
Exercises	158
Chapter 8. Local Classification of Lattices	161
§8.1. Jordan Splittings	161
§8.2. Nondyadic Classification	164
§8.3. Towards 2-adic Classification	165
Exercises	171
Chapter 9. The Local-Global Approach to Lattices	175
§9.1. Localization	176
§9.2. The Genus	178
§9.3. Maximal Lattices and the Cassels–Pfister Theorem	181
§9.4. Sums of Integer Squares (Second Version)	184
Exercises	187
$9.5.$ Indefinite Unimodular $\mathbb{Z}$ -Lattices	188
§9.6. The Eichler–Kneser Theorem; the Lattice $\mathbb{Z}^n$	191
§9.7. Growth of Class Numbers with Rank	196
§9.8. Introduction to Neighbor Lattices	201
Exercises	205
Chapter 10. Lattices over $\mathbb{F}_q[x]$	207
§10.1. An Initial Example	209
§10.2. Classification of Definite $\mathbb{F}_q[x]$ -Lattices	210
§10.3. On the Hasse–Minkowski Theorem over $\mathbb{F}_q(x)$	218
§10.4. Representation by $\mathbb{F}_q[x]$ -Lattices	220
Exercises	223
Chapter 11. Applications to Cryptography	225
§11.1. A Brief Sketch of the Cryptographic Setting	225
§11.2. Lattices in $\mathbb{R}^n$	227

$\S{11.3.}$	LLL-Reduction	230
§11.4.	Lattice Attacks on Knapsack Cryptosystems	235
$\S{11.5.}$	Remarks on Lattice-Based Cryptosystems	239
Appendix: Further Reading		
Bibliography		