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Edited by P. R. Halmos

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Claude George

Exercises in Integration

With 6 Illustrations

Springer-Verlag
New York Berlin Heidelberg Tokyo

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AMS Classifications: 00A07, 26-01, 28-01

Library of Congress Cataloging in Publication Data
George, Claude.
Exercises in integration.
(Problem books in mathematics)
Translation of: Exercices et problèmes d'intégration.
Bibliography: p.
Includes indexes.
1. Integrals, Generalized–Problems, exercises,
etc. I. Title. II. Series.
QA312.G39513 1984 515.4 84-14036

Title of the original French edition: *Exercices et problèmes d'intégration,*
© BORDAS, Paris, 1980.

9 8 7 6 5 4 3 2 1

ISBN-13: 978-1-4684-6322-4 e-ISBN-13: 978-1-4684-6320-0
DOI: 10.1007/978-1-4684-6320-0

Introduction

Having taught the theory of integration for several years at the University of Nancy I, then at the École des Mines of the same city, I had followed the custom of the times of writing up detailed solutions of exercises and problems, which I used to distribute to the students every week. Some colleagues who had had occasion to use these solutions have persuaded me that this work would be interesting to many students, teachers and researchers. The majority of these exercises are at the master's level; to them I have added a number directed to those who would wish to tackle greater difficulties or complete their knowledge on various points of the theory (third year students, diploma of education students, researchers, etc.).

This book, I hope, will render to students the services that this kind of book brings them in general, with the reservation that can always be made in this case: that certain of them will be tempted to look at the solution to the exercises which are put to them without any personal effort. There is hardly any need to emphasize that such a use of this book would be no benefit. On the other hand, the student who after having worked seriously upon a problem, seeks some pointers from the solution, or compares it with his own, will be using this work in the optimal way.

Teachers will find this book to be an important, if not exhaustive, list of exercises, certain of which are more or less standard, and others which may seem new.

I have also noted (and this is what led me to edit these sheets) that from one year to another one sometimes forgets the solution of an exercise and that one has to lose precious time in rediscovering it. This is particularly true for those solutions of which one remembers the heuristic form but of which the writing up is delicate if one wishes to be clear and precise at the same time. Now, if one requires, quite rightly, that students write their homework up correctly, then it is befitting to submit impeccable corrections to them, where the notations are judiciously chosen, phrases of the kind "it is clear that ... " used wittingly, and where the telegraphic style gives way to conciseness. It is often the incorporation of these corrections which demands the most work; I have therefore striven to take pains with the preparation of the proposed solutions, always remaining persuaded that perfection in this domain is never attained. If this book encourages those who have to present (either orally or in writing) correct versions of problems to improve the version they submit, the object I have set myself will be partly realised.

In this book researchers will find some results that are not always treated in courses on integration; they are either properties whose use is not as universal as those which usually appear and which are therefore found scattered about in appendices in various works, or are results that correspond to some technical lemmas which I have picked up in recent articles on a variety of subjects: group theory, differential games, control theory, probability, etc., ...

In presenting such a work it is just as well to make explicit those points of the theory that are assumed to be known. This is the object of the brief outline which precedes the eleven chapters of exercises.

In view of the origin of this book, it is evident that I took as a reference point the course that I gave at the time. After having taught abstract measure theory one year, I opted the next for a course expounding only the Lebesgue integral. This is not the place to discuss the advantages and inconveniences of each of the two points of view for the first year of a master's programme. I will say only that I have always considered the course that I gave to be more a course in analysis in which it is decided to use the Lebesgue integral than as a dogmatic exposition of a particular theory of integration. The choice of exercise reflects this attitude, especially in the emphasis given to trigonometric series, thereby paying the hommage due to the theory which is the starting point of the works of Cantor, Jordan, Peano, Borel, and Lebesgue. From this it results that, except for the seven exercises of Chapter 2 concerning σ-algebras, all the others deal with Lebesgue measure on $\mathbb{R}^n$. The advantage that has to be conceded to this point of view is that it avoids the vocabulary of abstract measure theory, which constitutes an artificial obstacle for those readers who might not yet be well versed in this theory. As for students who might have followed a more sophisticated course, I can assure them that by substituting $d\mu$ for dx and $\mu(E)$ fore meas(E) they will essentially rediscover the problems as they are commonly put to them, except for pathological examples about measures that are not σ-finite and the applications of the Radon-Nikodym Theorem. Furthermore, on this latter point the more perspicacious amongst them will not fail to see that the chapter treating the relationships between differentiation and integration is not foreign to this theorem. Truthfully, there is another point that is not tackled in this book, namely the matter of Fourier transforms of finite positive measures and Stone's Theorem, which to my mind is better suited to a course on probability.

As was mentioned above, numerous exercises are devoted to trigonometric series, which provides an important set of applications

of Lebesgue's theory. This has led me to include some exercises on series, summation processes, and trigonometric polynomials.

Other exercises use the theory of holomorphic functions. In particular, some results of the Phrägmen-Lindelöf type arise on two occasions; in each instance I have given its proof under the hypotheses that appear in the exercise. Quite generally, I have included in the solutions, or in an appendix to them, the proofs of certain points of analyis, topology, or algebra which students may not know.

I have chosen to make each solution follow immediately after the corresponding problem. The other method, which consists of regrouping the former in a second part of the work, seemed to me (from memories I have retained from my student days) much less manageable, especially when the problem is long, for it then becomes necessary to return often to the back of the book in order to follow the solution.

I find it difficult to cite the origin of these exercises. Many are part of a common pool of knowledge, handed down, one might say, in the public domain. Others are drawn from different classic works where they are proposed without proof or followed by more or less summary indications (in this respect it is interesting to note that in forcing oneself to write down the solutions one discovers a certain number of errors—just as many in the questions as in the suggestions offered). Certain of the exercises in this book were communicated to me orally by colleagues; I would thank them for their help here. Lastly, others are, as I have already said, lemmas found here and there, and which I have sometimes adapted.

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