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Methods of Homological Algebra

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Foreword

... utinam intelligere possim rationacinationes pulcherrimas quae e propositione concisa DE QUADRATUM NIHILO EXAEQUARI fluunt.

(... if I could only understand the beautiful consequence following from the concise proposition $d^2 = 0$.)

From Henri Cartan Laudatio on receiving the degree of Doctor Honoris Causa, Oxford University, 1980

1

Homological algebra first arose as a language for describing topological properties of geometrical objects. The emergence of a new language is always an important event in the development of mathematics: Euclidean plane and spatial geometry, Cartesian analytic geometry, the formalization of Newton's fluents and fluxions by Leibniz and later by Lagrange start the series to which homological algebra can be added. As with every successful language, homological algebra quickly realized its tendencies for self-development. As with every successful mathematical language, it rapidly began to expand its semantics, that is, to describe things that it was not originally designed to describe. The computation of the index of an elliptic operator, exact estimates for the number of solutions of congruences modulo a prime, the theory of hyperfunctions, anomalies in quantum field theory – these are only some of the contemporary applications of homological ideas.

The history of homological algebra can be divided into three periods. The first one starts in the 1940's with the classical works of Eilenberg and MacLane, D.K. Faddeev, and R. Baer and ends with the appearance in 1956 of the fundamental monograph "Homological Algebra" by Cartan and Eilenberg which has lost none of its significance up to the present day.

A. Grothendieck's long paper "Sur quelques points d'algèbre homologique" published in 1957 (its appearance had been delayed three years) marks the starting point of the second period, which was dominated by the influence of Grothendieck and his school of algebraic geometry.

The third period, which extends up to the present time, is marked by the ever-increasing use of derived categories and triangulated categories. The basic technique was developed in the thesis of Grothendieck's student J.-L. Verdier in 1963, but was slow in spreading beyond the confines of algebraic geometry. Only in the last fifteen years has the situation changed. First in the work of M. Sato and his school on microlocal analysis, then in the theory of D-modules and perverse sheaves with applications to representation theory, derived categories started to be used as the most suitable instrument.

We now try to characterize these three periods, although we should apologize to the reader for our subjective evaluation and judgment and for the incompleteness of the material: of course, many important developments do not fit into our rigid scheme.

The book by Cartan and Eilenberg contains essentially all the constructions of homological algebra that constitute its computational tools, namely standard resolutions and spectral sequences. No less important, it contains an axiomatic definition of derived functors of additive functors on the category of modules over a ring.

It was this idea that determined the contours of the second period. The logic of the internal development of analytic and algebraic geometry led to the formulation of the notion of a sheaf and to the realization of the idea that the natural argument of a homology theory is a pair consisting of a space with a sheaf on it, rather than just a space (or a space and a coefficient group). Here the fundamental contribution of H. Cartan's seminar and J.-P. Serre's paper "Faisceaux algébriques cohérents" should be mentioned. Grothendieck's paper of 1957 quoted above stresses the analogy between pairs (space, sheaf of abelian groups on it) and pairs (ring, module over it) from the homological point of view and emphasizes the idea that sheaf cohomology should be defined as the derived functor of global sections.

The break with the axiomatic homology and cohomology theory of Eilenberg and Steenrod is in that now an abelian object (a sheaf), rather than a non-abelian one (a space), serves as a variable argument in a cohomology theory. More precisely, a homology or a cohomology theory with fixed coefficients according to Eilenberg and Steenrod is a graded functor from the category of topological spaces into abelian groups that satisfies certain axioms by which it is uniquely determined. The most important of these axioms are the specification of the homology (or cohomology) of the point, and the exact sequence associated with the "excision axiom". The cohomology theory of a fixed topological space according to Grothendieck is a graded functor from the category of sheaves of abelian groups on this space into abelian groups, also satisfying a number of axioms by which it is uniquely determined. The most important of these are the specification of zero-dimensional cohomology as global sections and the exact sequence associated with a short exact sequence of sheaves.

The development of this idea led to a very far-reaching generalization of basic notions of algebraic geometry – Grothendieck topologies and topoi. The essence of this generalization is that since the cohomological properties of a space are completely determined by the category of sheaves over it, it is these categories that should be the primary objects of study in topology, rather than topological spaces themselves. After a suitable axiomatization of the properties of such categories we arrive at the notion of a topos. The development of these abstract ideas was motivated by a very concrete problem – the famous conjectures of A. Weil on the number of solutions of congruences modulo a prime. The very statements of these conjectures include the assumption about the existence of a certain cohomology theory of algebraic varieties in characteristic $p > 0$, which would allow us to apply to this situation the Lefschetz fixed point formula; a cohomology theory of this type was provided by the cohomology of the etale topos constructed by A. Grothendieck and developed by his students.

The main product of the homological algebra of this period was the computation and properties of various derived functors R^pF, where F is the functor of global sections, of direct image, of tensor product and so on. These derived functors arise as the cohomology of complexes of the form $F(I^\cdot)$, where $I^\cdot$ are resolutions consisting of injective, projective, flat, or some other objects suitably adapted to F. The choice of a resolution is highly non-unique, but R^pF does not depend on this choice.

In the course of time it came to be understood that one should study all complexes, rather that just resolutions $I^\cdot$ (and complexes obtained by applying functors to these resolutions), but modulo a quite complicated equivalence relation, which identifies certain complexes having the same cohomology.

The final version of this equivalence relation seems still not to be completely understood. However, a working definition which has proved its worth was formulated in Verdier's thesis of 1963. The categories of complexes obtained in this way are called derived categories, and axiomatization of their properties leads to the notion of triangulated categories.

It seems to us that the main feature of the third period of homological algebra is the development of a special kind of "thinking in terms of complexes" as opposed to the "thinking in terms of objects and their cohomological invariants" that was typical for the first two periods. Perhaps this appears most vividly in the theory of perverse sheaves; it was shown that the cohomological properties of topological manifolds extend to a substantial degree to spaces with singularities, if we take as coefficients not sheaves but special complexes of sheaves (as objects of the corresponding derived category). The conormal complexes of Grothendieck and Illusie and the dualizing complexes of Grothendieck and Verdier can be considered as earlier constructions of the same kind.

2

This book is intended as an introductory textbook on the technique of derived categories. Up to now, as far as we know, a mathematician willing to learn this subject has had to turn either to the two original sources, the abstract of Verdier's thesis and the notes of Hartshorne's seminar, or to the oral tradition, in those mathematical centers where it still has been maintained.

Thus the central part of the book is Chaps. III–IV, and the reader with even a slight acquaintance with abelian categories and functors can start directly from Chap. III.

Chapter II is directed to the reader who has hardly had anything to do with categories, and we have tried to make clear the intuitive meaning of standard categorical constructions, and to give examples of "thinking in categories". The main practical aim of this chapter is an introduction to abelian categories.

Finally, Chaps. I and V resulted from our attempt (which had cost us a lot of trouble) to separate off homological algebra from algebraic topology, without burning the bridge between them. Triangulated spaces and simplicial sets are perhaps the most direct methods of describing topology in terms of algebra, and we decided to start the book with an introduction to simplicial methods. On the other hand, algebraic topology is unthinkable without homotopy theory, and the book ends with a treatment of the foundations of homotopic algebra in Chap. V.

We worked on this book with the disquieting feeling that the development of homological algebra is currently in a state of flux, and that the basic definitions and constructions of the theory of triangulated categories, despite their widespread use, are of only preliminary nature (this applies even more to homotopic algebra). There is no doubt that similar thoughts have occurred to the founders of the theory, and to everyone who has seriously worked with it; the absence of a monographic exposition is one of the symptoms.

Nevertheless, this period has already lasted twenty years; papers whose main results cannot even be stated in the old language are multiplying; the need for a textbook is growing. We therefore present this book to the benevolent judgment of the reader.

3

The plan of the book evolved gradually over several years when the authors were running seminars in the Mathematics Department of Moscow University, and were in contact with members of the "Homological Algebra Fan Club". A.A. Beilinson, M.M. Kapranov, V.V. Schechtman, whose papers and explanations provided us with live examples of thinking in complexes.

J.-P. Serre, J.N. Bernstein and M.M. Kapranov have read the manuscript and made a series of very useful comments.

V.E. Govorov very kindly to provided us with an extensive card index of works on homological algebra.

We are grateful to all them, and also to V.A. Ginzburg, R. MacPherson, S.M. Khoroshkin and B.L. Tsygan.

Our debt to the founding fathers of the subject, whose books, papers and ideas we have used and have been inspired by, should be obvious from the contents.

Moscow, 1988 S.I. Gelfand, Yu.I. Manin

Reference Guide

1. General References. Five main sources for the classical homological algebra are books by Cartan – Eilenberg [1], MacLane [1], Hilton – Stammbach [1], Bourbaki [1] and the large paper by Grothendieck [1]. Simplicial methods are presented in Gabriel – Zisman [1] and in May [1], sheaves in Godement [1], Bredon [1], Golovin [1], Iversen [1]. Topoi are discussed, in particular, in Goldblatt [1] and Johnstone [1]. Among the books on cohomology of various algebraic structures we mention Brown [1], Serre [8], Guichardet [1], Fukchs [2]. A large list of books on algebraic topology contains, among others, Eilenberg – Steenrod [1], Hilton – Wiley [1], Spanier [1], Dold [1], Massey [2], Boardman – Vogt [1], Fukchs [1], Dubrovin – Novikov – Fomenko [1], Bott – Tu [1].

Modern algebraic geometry is an ample source of homological algebra of various kind. Here we must mention the pioneering paper by Serre [3] and the publications of Grothendieck and his school: Grothendieck – Dieudonné [1] (especially Chaps. 0 and III) and [2], Grothendieck et al. [SGA] (especially 4, 4 1/2, 6), Artin [1], Hartshorne [1], Berthelot [1], Deligne [1], [2]. Among several textbooks on this subject we mention Hartshorne [2] and Milne [1].

The history of the homological algebra has yet to be written; we can recommend to the interested reader the paper by Grey [1], the corresponding parts from Dieudonné [1] and reminiscences of Grothendieck [5].

2. Topics We Have Not Considered in the Book.

a) Noncommutative cohomology. Some problems in group theory and topology lead to cohomology with non-commutative coefficients. A systematic theory exists only in low dimensions (≤ 2 or ≤ 3). An excellent exposition for the case of group cohomology based on the paper by Dedecker [1] is Serre [7]. Most commonly used is 1-cohomology, or torsors. About intermediate "state of the art" see Giraud [1].

b) Derivatives of non-additive functors. First constructions of derivatives of non-additive functors, such as the symmetric or the exterior power of a module, were suggested by Dold – Puppe [1]. Their technique was developed further by Illusie [1] who applied such functors in certain algebraic geometry situations. Crucial in the construction of these functors are simplicial methods. In Feigin – Tsygan [1], [2] the additive K-theory is interpreted as the derivation of the functor that associates to each ring its quotient by the commutant.

c) Continuous cohomology. Functional analysis and infinite-dimensional geometry produce some cohomology-like construction in various categories of algebraic structures with topology, such as linear topological spaces, Banach algebras, Lie groups, etc. However, most of these categories (and the most important ones) are non-abelian, and the standard technique does not work. Usually in definition and computations the authors exploit some specific classes of complexes. See Helemski [1], Guichardet [1], Borel – Wallach [1], Johnson [1].

d) Products and duality. Some odds and ends the reader can find in various parts of the book, but a satisfactory general theory in the framework of homological algebra presumably does not exist. See [SGA 2] and Hartshorne [1] about the duality in algebraic geometry, Verdier [1], [2] and Iversen [1] about the duality in topology. The theory of DG-algebras (see Chap. V) can be considered as an attempt to introduce the multiplicative structure "from scratch". About deeper results see Boardman – Vogt [1], Shechtman [1], Hinich – Shechtman [1]. Classical theory of cohomological operations (Steenrod powers, Massey operations) also can be considered from such viewpoint.

e) Homological algebra and K-theory. The literature on K-theory is very ample; see the basic papers by Quillen [1], [2], [4], the review by Suslin [2], as well as [KT1], [KT2], where one can find further references.

f) Miscelleneous. Applications of Galois cohomology in number theory are based, first of all, on class field theory; see the classical exposition in Artin – Tate [1] and subsequent papers by Tate [1], Mazur [1], among others. There exists a large literature on homological methods in commutative algebra; see [ATT], Serre [6], André [2], [3], Avramov – Halperin [1], Quillen [3]. About some other applications of homological algebra see [AN], [ES], [SD].

3. To Chapter I. Sect. I.1–I.3: About further results of simplicial algebra, and, in particular, about its applications to homological algebra, see Gabriel – Zisman [1] and May [1]; see also the remarks to Chap. V below. Its application to the derivation of non-additive functors see in Dold – Puppe [1] and Illusie [1]. Deligne extensively used symplicial methods in the theory of mixed Hodge structures, see Deligne [1], Beilinson [2]. About Exercises 2, 3 to Sect. I.2 see Duskin [1], [2].

Sect. I.4: Algebraic topology is only slightly mentioned here, see Sect. 1 of this guide.

Sect. I.5: About the classical sheaf theory see Serre [3], [4], Godement [1], Bredon [1], Golovin [1], Iversen [1]. For sheaf theory in general topoi, as well as in ètale, cristalline, and other topoi of algebraic geometry see [SGA 4], Artin [1], Berthelot [1], Milne [1].

The most important development of the sheaf theory in the last ten years is related to the notion of a perverse sheaf and the corresponding cohomological formalism which is well suited to the study of singular varities. Perverse sheaves are objects of the derived category of usual sheaves, so that a perverse sheaf is a complex of usual sheaves. See Goresky – MacPherson [1], Beilinson – Bernstein – Deligne [1], and [IH], [ES].

Sect. I.6: An exact sequence is the main tool of homological algebra. See further development in the framework of derived and triangulated categories in Sect. III.3 and IV.1.

Sect. I.7: There exists a large list of papers whose authors introduce and study important specific resolution and complexes such as de Rham, Čech, Koszul, Hochshild, and bar-resolutions, cyclic complexes, complexes of continuous cochains, etc. See, in particular, Priddy [1], [2], Karoubi [1], [2], [3], Connes [1], Fukchs [2], Hochshild [1].

4. To Chapter II. Sect. II.1: See MacLane [1], Goldblatt [1], Faith [1]; about 2-categories see Gabriel – Zisman [1].

Sect. II.2: About the theory of fundamental group in algebraic geometry see [SGA 2], about Gelfand duality see Gelfand – Shilov [1], about Morita equivalence see Morita [1], Faith [1]. A classical example of the non-trivial equivalence is the description of coherent sheaves on projective algebraic manifolds by corresponding modules over homogeneous coordinate ring, see Serre [3] and a generalization in Grothendieck – Dieudonné [1, EGA 3].

Further generalizations of these ideas lead to remarkable equivalences between some derived categories, see Sect. IV.3.

Sect. II.3: Several important theorems give an abstract characterization of representable functors. About Freyd's theorem in the general category theory framework see MacLane [2]. A lot of important spaces (like moduli spaces, i.e. bases of universal deformations) in algebraic and analytic geometry are introduced using the notion of a representable functor. In this context the characterization of representable functors by a short list of easily verified properties leads to some fundamental existence theorems, see Grothendieck [1], [2], [4], Artin [1], Knutson [1].

The fundamental notion of the adjoint functor was introduced by Kan [1]. Several important constructions in algebra, geometry, and topology can be described using this notion, see examples in André [1], Faith [1], MacLane [2].

Sect. II.4: For details about ringed spaces see Grothendieck – Dieudonné [1, Chap. 0], [2]. For the nerve of a category see Quillen [4], Suslin [1]. For quadratic algebras (Ex. 5) see Manin [1].

Sect. II.5–II.6: This is a classical part of the theory of abelian categories, see Cartan – Eilenberg [1], MacLane [1], [2], Grothendieck [1]. For the development of these ideas in the context of derived categories see Sect. III.6, IV.1. About ex. 1–7 in Sect. II.5 see MacLane [2]. About ex. 9 in Sect. II.5 see Serre [2].

5. To Chapter III. Sect. III.1–III.4: See Hartshorne [1], Verdier [3]. The fundamental diagram in Lemma III.3.3 is taken from Bourbaki [1].

It seems that the main deficiency of the definition of a derived category is in the bad definition of distinguished triangles. The problem of what should be a good definition is discussed in unpublished notes of Deligne. See also the discussion about the functor det in Knudsen – Mumford [1], and in [SGA 6] and the definition of Tot in Exercises to IV.2.

Sect. III.5: The classical theory of the functors Ext in terms of complexes is due to Yoneda [1] (it generalizes the Baer's theory of Ext^1). Homological dimension was studied in algebraic geometry (Serre [4]), in depth theory ([SHA 2]), in group theory (Brown [1] and several papers in [HG]).

About theorem III.5.21 see Hartshorne [1]. This theorem can be considered as one of the theorems establishing the equivalence between a derived category and a category of complexes modulo homotopic equivalence, see Beilinson [1], Bernstein – Gelfand – Gelfand [1], Kapranov [1], [2], [3]. For results related to ex. 4 see Happel [1].

Sect. III.6: The main references here are the same as in Sects. III.1–III.4. For ex. 1–5 see Deligne's paper in Grothendieck et al. [SGA 4, XVII], for ex. 6 see Roos [1], [2], about ex. 7–10 see Spaltenstein [1].

Sect. III.7: While an exact sequence can be considered as the main tool in the study of cohomology dependence on the abelian variable, a spectral sequence plays a similar role in the study of the dependence on the non-abelian variable. The first spectral sequence was introduced, presumably, by Leray [1]; the classical exposition of Serre [1] remains an excellent introduction into the subject. The standard construction of the spectral sequence associated to a filtered complex is given in Cartan – Eilenberg [1], and the one associated to an exact couple is given in Massey [1] (see also Eckmann – Hilton [1], [2]). Grothendieck [1] showed that some standard spectral sequences relate derived functors of the composition to the derived functors of factors. However, spectral sequences in homotopic topology are of different nature, see McCleary [1]. Fukchs [1] gives a fascinating description of the Adams spectral sequence. See also exercises to IV.2.

Sect. III.8: This section, together with exercises to it and to IV.4, presents sheaf cohomology theory, as it is seen nowadays. The main difference from the status fixed in Godement [1] is the appearence of the functor $f^!$, which can be defined only using derived categories. This functor leads to the Verdier duality, which also can be formulated only in derived categories, and to the "six functors" formalism (see exercises to IV.4). References are Verdier [1], [2], [4], the volume [IH], Iversen [1]. The parallel theory in algebraic geometry is presented in Hartshorne [1] for coherent sheaves and in [SGA 4] (especially XVII) for étale topology.

6. To Chapter IV. Sect. IV.1–IV.2: The main sources for us were Verdier [3], Hartshorne [2] and Kapranov [3]. See also Happel [1], Iversen [1]. Exercises to Sect. IV.2 were composed by Kapranov.

Sect. IV.3: The description of derived categories of coherent sheaves on on projective spaces was initiated in Beilinson [1] and Bernstein – Gelfand –

Gelfand [1]. A series of consecutive generalizations of this theory was found by Kapranov [1], [2], [3]; see also Meltzer [1]. The "$S - \Lambda$ duality" described in this section possesses far reaching generalizations, see Priddy 1], [2], Lofwall [1], Happel [1], Gorodentsev – Rudakov [1].

Sect. IV.4: We present here some ideas from Beilinson – Bernstein – Deligne [1]. The main application of this theory, that is, the construction of perverse sheaves, is not discussed in this book. About ex. 1–5 see Beilinson – Bernstein – Deligne [1], about ex. 6 see Bernstein – Gelfand – Ponomarev [1], Brenner – Butler [1], about ex. 7 see Happel [1].

7. To Chapter V. In this chapter we study homotopic algebra (algebraic foundations of homotopy theory), which much less developed than homological algebra.

Sect. V.1–V.2: Here we introduce the main axiomatic notion that of a closed model category (Quillen [1]), which axiomatizes the main homotopic properties of topological spaces. Since we view simplicial sets as a bridge between topology and algebra, we give a (rather lengthy) proof that simplicial sets form a closed model category. We hope that these two sections will help an interested reader to study deeper parts of the book by Quillen [1], as well as further literature: Quillen [2], [4], May [1], Bousfield – Gugenheim [1], Tanré [1].

Sect. V.3–V.4: The second part of this chapter introduces to the reader some ideas of the famous paper by Sullivan [1] where he shows that the rational homotopic type of a manifold can be determined by its algebra of differential forms.

We prove that differential graded algebras form a closed model category and study minimal models in this category.

Exercises to these sections are based, mainly, on results from Tanré [1].

Sect. V.5: Here we present (without proofs) main results of the theory of rational homotopic type. The proofs, together with further details and references, can be found in Lemann [1], Bousfield – Gugenheim [1], Deligne – Griffiths – Morgan – Sullivan [1], Morgan [1], Halperin [1], Avramov – Halperin [1], Tanré [1].

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