

# Contents

|   |             |    |
|---|-------------|----|
| <i>Introduction</i>                                   | <i>page</i> | 1  |
| <b>1 Measure and integral</b>                         |             | 4  |
| 1.1 Measure   |             | 4  |
| 1.2 Measurable functions                              |             | 7  |
| 1.3 Integration                                       |             | 9  |
| 1.4 Notes and remarks                                 |             | 12 |
| <b>2 The Cauchy–Schwarz inequality</b>                |             | 13 |
| 2.1 Cauchy’s inequality                               |             | 13 |
| 2.2 Inner-product spaces                              |             | 14 |
| 2.3 The Cauchy–Schwarz inequality                     |             | 15 |
| 2.4 Notes and remarks                                 |             | 17 |
| <b>3 The AM–GM inequality</b>                         |             | 19 |
| 3.1 The AM–GM inequality                              |             | 19 |
| 3.2 Applications                                      |             | 21 |
| 3.3 Notes and remarks                                 |             | 23 |
| <b>4 Convexity, and Jensen’s inequality</b>           |             | 24 |
| 4.1 Convex sets and convex functions                  |             | 24 |
| 4.2 Convex functions on an interval                   |             | 26 |
| 4.3 Directional derivatives and sublinear functionals |             | 29 |
| 4.4 The Hahn–Banach theorem                           |             | 31 |
| 4.5 Normed spaces, Banach spaces and Hilbert space    |             | 34 |
| 4.6 The Hahn–Banach theorem for normed spaces         |             | 36 |
| 4.7 Barycentres and weak integrals                    |             | 39 |
| 4.8 Notes and remarks                                 |             | 40 |
| <b>5 The <math>L^p</math> spaces</b>                  |             | 45 |
| 5.1 $L^p$ spaces, and Minkowski’s inequality          |             | 45 |

|          |   |            |
|----------|---|------------|
| 5.2      | The Lebesgue decomposition theorem                | 47         |
| 5.3      | The reverse Minkowski inequality                  | 49         |
| 5.4      | Hölder's inequality                               | 50         |
| 5.5      | The inequalities of Liapounov and Littlewood      | 54         |
| 5.6      | Duality   | 55         |
| 5.7      | The Loomis–Whitney inequality                     | 57         |
| 5.8      | A Sobolev inequality                              | 60         |
| 5.9      | Schur's theorem and Schur's test                  | 62         |
| 5.10     | Hilbert's absolute inequality                     | 65         |
| 5.11     | Notes and remarks                                 | 67         |
| <b>6</b> | <b>Banach function spaces</b>                     | <b>70</b>  |
| 6.1      | Banach function spaces                            | 70         |
| 6.2      | Function space duality                            | 72         |
| 6.3      | Orlicz spaces                                     | 73         |
| 6.4      | Notes and remarks                                 | 76         |
| <b>7</b> | <b>Rearrangements</b>                             | <b>78</b>  |
| 7.1      | Decreasing rearrangements                         | 78         |
| 7.2      | Rearrangement-invariant Banach function spaces    | 80         |
| 7.3      | Muirhead's maximal function                       | 81         |
| 7.4      | Majorization                                      | 84         |
| 7.5      | Calderón's interpolation theorem and its converse | 88         |
| 7.6      | Symmetric Banach sequence spaces                  | 91         |
| 7.7      | The method of transference                        | 93         |
| 7.8      | Finite doubly stochastic matrices                 | 97         |
| 7.9      | Schur convexity                                   | 98         |
| 7.10     | Notes and remarks                                 | 100        |
| <b>8</b> | <b>Maximal inequalities</b>                       | <b>103</b> |
| 8.1      | The Hardy–Riesz inequality ( $1 < p < \infty$ )   | 103        |
| 8.2      | The Hardy–Riesz inequality ( $p = 1$ )            | 105        |
| 8.3      | Related inequalities                              | 106        |
| 8.4      | Strong type and weak type                         | 108        |
| 8.5      | Riesz weak type                                   | 111        |
| 8.6      | Hardy, Littlewood, and a batsman's averages       | 112        |
| 8.7      | Riesz's sunrise lemma                             | 114        |
| 8.8      | Differentiation almost everywhere                 | 117        |
| 8.9      | Maximal operators in higher dimensions            | 118        |
| 8.10     | The Lebesgue density theorem                      | 121        |
| 8.11     | Convolution kernels                               | 121        |
| 8.12     | Hedberg's inequality                              | 125        |

|           |  |     |
|-----------|--|-----|
| 8.13      | Martingales  | 127 |
| 8.14      | Doob's inequality  | 130 |
| 8.15      | The martingale convergence theorem   | 130 |
| 8.16      | Notes and remarks  | 133 |
| <b>9</b>  | <b>Complex interpolation</b>   | 135 |
| 9.1       | Hadamard's three lines inequality  | 135 |
| 9.2       | Compatible couples and intermediate spaces                                 | 136 |
| 9.3       | The Riesz–Thorin interpolation theorem                                     | 138 |
| 9.4       | Young's inequality   | 140 |
| 9.5       | The Hausdorff–Young inequality   | 141 |
| 9.6       | Fourier type   | 143 |
| 9.7       | The generalized Clarkson inequalities                                      | 145 |
| 9.8       | Uniform convexity  | 147 |
| 9.9       | Notes and remarks  | 150 |
| <b>10</b> | <b>Real interpolation</b>  | 154 |
| 10.1      | The Marcinkiewicz interpolation theorem: I                                 | 154 |
| 10.2      | Lorentz spaces   | 156 |
| 10.3      | Hardy's inequality   | 158 |
| 10.4      | The scale of Lorentz spaces  | 159 |
| 10.5      | The Marcinkiewicz interpolation theorem: II                                | 162 |
| 10.6      | Notes and remarks  | 165 |
| <b>11</b> | <b>The Hilbert transform, and Hilbert's inequalities</b>                   | 167 |
| 11.1      | The conjugate Poisson kernel   | 167 |
| 11.2      | The Hilbert transform on $L^2(\mathbf{R})$                                 | 168 |
| 11.3      | The Hilbert transform on $L^p(\mathbf{R})$ for $1 < p < \infty$            | 170 |
| 11.4      | Hilbert's inequality for sequences   | 174 |
| 11.5      | The Hilbert transform on $\mathbf{T}$                                      | 175 |
| 11.6      | Multipliers  | 179 |
| 11.7      | Singular integral operators  | 180 |
| 11.8      | Singular integral operators on $L^p(\mathbf{R}^d)$ for $1 \leq p < \infty$ | 183 |
| 11.9      | Notes and remarks  | 185 |
| <b>12</b> | <b>Khinchine's inequality</b>  | 187 |
| 12.1      | The contraction principle  | 187 |
| 12.2      | The reflection principle, and Lévy's inequalities                          | 189 |
| 12.3      | Khinchine's inequality   | 192 |
| 12.4      | The law of the iterated logarithm  | 194 |
| 12.5      | Strongly embedded subspaces  | 196 |
| 12.6      | Stable random variables  | 198 |
| 12.7      | Sub-Gaussian random variables  | 199 |

|           |  |            |
|-----------|--|------------|
| 12.8      | Kahane's theorem and Kahane's inequality                     | 201        |
| 12.9      | Notes and remarks  | 204        |
| <b>13</b> | <b>Hypercontractive and logarithmic Sobolev inequalities</b> | <b>206</b> |
| 13.1      | Bonami's inequality  | 206        |
| 13.2      | Kahane's inequality revisited                                | 210        |
| 13.3      | The theorem of Latała and Oleszkiewicz                       | 211        |
| 13.4      | The logarithmic Sobolev inequality on $D_2^d$                | 213        |
| 13.5      | Gaussian measure and the Hermite polynomials                 | 216        |
| 13.6      | The central limit theorem                                    | 219        |
| 13.7      | The Gaussian hypercontractive inequality                     | 221        |
| 13.8      | Correlated Gaussian random variables                         | 223        |
| 13.9      | The Gaussian logarithmic Sobolev inequality                  | 225        |
| 13.10     | The logarithmic Sobolev inequality in higher dimensions      | 227        |
| 13.11     | Beckner's inequality   | 229        |
| 13.12     | The Babenko–Beckner inequality                               | 230        |
| 13.13     | Notes and remarks  | 232        |
| <b>14</b> | <b>Hadamard's inequality</b>                                 | <b>233</b> |
| 14.1      | Hadamard's inequality  | 233        |
| 14.2      | Hadamard numbers   | 234        |
| 14.3      | Error-correcting codes                                       | 237        |
| 14.4      | Note and remark  | 238        |
| <b>15</b> | <b>Hilbert space operator inequalities</b>                   | <b>239</b> |
| 15.1      | Jordan normal form   | 239        |
| 15.2      | Riesz operators  | 240        |
| 15.3      | Related operators  | 241        |
| 15.4      | Compact operators  | 242        |
| 15.5      | Positive compact operators                                   | 243        |
| 15.6      | Compact operators between Hilbert spaces                     | 245        |
| 15.7      | Singular numbers, and the Rayleigh–Ritz minimax formula      | 246        |
| 15.8      | Weyl's inequality and Horn's inequality                      | 247        |
| 15.9      | Ky Fan's inequality  | 250        |
| 15.10     | Operator ideals  | 251        |
| 15.11     | The Hilbert–Schmidt class                                    | 253        |
| 15.12     | The trace class  | 256        |
| 15.13     | Lidskii's trace formula                                      | 257        |
| 15.14     | Operator ideal duality                                       | 260        |
| 15.15     | Notes and remarks  | 261        |
| <b>16</b> | <b>Summing operators</b>                                     | <b>263</b> |
| 16.1      | Unconditional convergence                                    | 263        |

|           |  |            |
|-----------|--|------------|
| 16.2      | Absolutely summing operators   | 265        |
| 16.3      | $(p, q)$ -summing operators  | 266        |
| 16.4      | Examples of $p$ -summing operators                                       | 269        |
| 16.5      | $(p, 2)$ -summing operators between Hilbert spaces                       | 271        |
| 16.6      | Positive operators on $L^1$  | 273        |
| 16.7      | Mercer's theorem   | 274        |
| 16.8      | $p$ -summing operators between Hilbert spaces ( $1 \leq p \leq 2$ )      | 276        |
| 16.9      | Pietsch's domination theorem   | 277        |
| 16.10     | Pietsch's factorization theorem  | 279        |
| 16.11     | $p$ -summing operators between Hilbert spaces ( $2 \leq p \leq \infty$ ) | 281        |
| 16.12     | The Dvoretzky–Rogers theorem   | 282        |
| 16.13     | Operators that factor through a Hilbert space                            | 284        |
| 16.14     | Notes and remarks  | 287        |
| <b>17</b> | <b>Approximation numbers and eigenvalues</b>                             | <b>289</b> |
| 17.1      | The approximation, Gelfand and Weyl numbers                              | 289        |
| 17.2      | Subadditive and submultiplicative properties                             | 291        |
| 17.3      | Pietsch's inequality   | 294        |
| 17.4      | Eigenvalues of $p$ -summing and $(p, 2)$ -summing endomorphisms          | 296        |
| 17.5      | Notes and remarks  | 301        |
| <b>18</b> | <b>Grothendieck's inequality, type and cotype</b>                        | <b>302</b> |
| 18.1      | Littlewood's $4/3$ inequality  | 302        |
| 18.2      | Grothendieck's inequality  | 304        |
| 18.3      | Grothendieck's theorem   | 306        |
| 18.4      | Another proof, using Paley's inequality                                  | 307        |
| 18.5      | The little Grothendieck theorem  | 310        |
| 18.6      | Type and cotype  | 312        |
| 18.7      | Gaussian type and cotype   | 314        |
| 18.8      | Type and cotype of $L^p$ spaces  | 316        |
| 18.9      | The little Grothendieck theorem revisited                                | 318        |
| 18.10     | More on cotype   | 320        |
| 18.11     | Notes and remarks  | 323        |
|           | <i>References</i>  | 325        |
|           | <i>Index of inequalities</i>   | 331        |
|           | <i>Index</i>   | 332        |

