R.V. Gamkrelidze (Ed.)

Geometry I

Basic Ideas and Concepts of Differential Geometry

With 62 Figures

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Basic Ideas and Concepts of Differential Geometry

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Translated from the Russian
by E. Primrose

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