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*(continued after index)*

Giovanni P. Galdi

An Introduction to the  
Mathematical Theory of the  
Navier-Stokes Equations

Volume I

Linearised Steady Problems  
Revised Edition



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*For Gino, Elena  
and  
Marisa*

# Preface

Undoubtedly, the Navier-Stokes equations are of basic importance within the context of modern theory of partial differential equations. Although the range of their applicability to concrete problems has now been clearly recognised to be limited, as my dear friend and bright colleague K.R. Rajagopal has showed me by several examples during the past six years, the mathematical questions that remain open are of such a fascinating and challenging nature that analysts and applied mathematicians cannot help being attracted by them and trying to contribute to their resolution. Thus, it is not a coincidence that over the past ten years more than seventy significant research papers have appeared concerning the well-posedness of boundary and initial-boundary value problems.

In this monograph I shall perform a systematic and up-to-date investigation of the fundamental properties of the Navier-Stokes equations, including existence, uniqueness, and regularity of solutions and, whenever the region of flow is unbounded, of their spatial asymptotic behavior. I shall omit other relevant topics like boundary layer theory, stability, bifurcation, detailed analysis of the behavior for large times, and free-boundary problems, which are to be considered "advanced" ones. In this sense the present work should be regarded as "introductory" to the matter.

I have divided the subject into two main parts: the first deals with steady flow (boundary value problem) and the second relates to time-dependent flow (initial-boundary value problem). Each part is self-contained and basically is not dependent on the other one, except for the case when the behavior in time of perturbations to a given steady motion will be investigated. In both parts the nonlinear analysis is always preceded and supported by a suitable linear analysis. This latter study, however, assumes an independent interest, and it is therefore treated in depth.

The *cantus firmus* of the entire work is represented by the approach in Lebesgue spaces  $L^q$ . In fact, this treatment is at the basis of the resolution of every problem considered throughout. I preferred to use such a general approach (instead of that in  $L^2$  or weighted  $L^2$ ), since it allows for a simpler and unified resolution of questions both in bounded and unbounded regions.

The book is essentially mathematically self-contained: the knowledge of Banach spaces and their basic properties (completeness, separability, reflexivity) along with some classical results on operator theory (e.g., contraction mapping theorem) are the only necessary prerequisites to reading this book, which is devoted to students (graduate and undergraduate) and those mathematicians and applied mathematicians who wish to become acquainted with the subject.

I would like to thank warmly all my colleagues, co-workers, and friends who, directly or indirectly, have contributed to the realisation of this book.

First of all, I am deeply indebted to Clifford Truesdell who in September 1983 invited me to write this monograph for the series *Springer Tracts in Natural Philosophy* and who, over the past several years, has constantly encouraged my efforts.

Dan Joseph, Salvatore Rionero, and Jim Serrin have independently and affectionately guided my early steps on the hard and nonetheless fascinating path of the Navier-Stokes theory, each one teaching me different perspectives. To all of them I am grateful.

I express my sincere gratitude to Christian Simader and Hermann Sohr, with whom I have enjoyed working and casting new light on more than one problem, and to Paolo Maremonti, Wolfgang Borchers, and Konstantin Pileckas, who are coauthors with me of several papers.

I also acknowledge, with pleasure, many stimulating and helpful conversations I had with V. Coscia, R. Farwig, V.N. Maslennikova, A. Novotný, M.C. Patria, R. Rautmann, R. Salvi, V.A. Solonnikov, W. von Wahl, and M. Wiegner. In particular, I wish to thank V. Coscia, A. Novotný, A. Passerini, and M.C. Patria for reading parts of the manuscript and pointing out to me various misprints and mistakes.

I have received great benefit from teaching a course at the winter school in Paseky (Czechoslovakia) during the month of December 1991, and having discussions with well-trained young mathematicians and their trainer, my friend Professor Enrico Nečas. For this (at least), I wish to thank the organisers of the school and, in particular, Josef Málek (Donna Peppa) and Michael Růžička.

Last, but not least, I wish to acknowledge the generous help of my dear friend Dr. Gino Valenti, Director of the Construction Management Division in the Office of the Architect of the Capitol, for allowing me access to the Library of Congress in Washington, D.C., during the academic year 1984-1985.

At this point, a married author usually recognises the unselfish and continuous encouragement of his wife. In fact, it is for me a great privilege and pleasure to thank my wife, Professor Mariarosaria Padula, for spending endless and enjoyable (to me) days and nights discussing and setting up every topic of the book. To her, who in every respect has to be considered coauthor of this monograph, I offer my everlasting gratitude and love.

Ferrara, via Porta Romana,  
June 1992

Giovanni Paolo Galdi

# Preface to the First Revised Edition

In this revised edition, I have corrected many misprints and some mistakes that were present in the first edition of the book. I wish to thank all colleagues who have kindly pointed them out to me. The list of their names is so long that if I would include it here, I would increase the probability of generating many other new misprints and mistakes, thus going through a neverending process.

There have been significant contributions to the mathematical theory of steady-state Navier-Stokes equations since the first edition of this book, like, for instance, those of K. Pileckas on the flow in domains with non-compact boundaries. These new achievements are reflected in the extended bibliography of this edition and referenced in the text. I have also entirely modified Section X.5 and, partly, Section X.6 to include the joint results of H. Sohr and me concerning the asymptotic structure of two-dimensional flow in exterior domains.

I completed this work during the academic year 1996/1997, while I was visiting the Department of Mathematics at the University of Pittsburgh. I take this opportunity to thank my colleagues for warm hospitality and stimulating discussions and, in particular, John Chadam, Bill Layton and Bryce McLeod.

Pittsburgh, Beechwood Blvd,  
April 1997

Giovanni Paolo Galdi



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