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# Fundamentals of Diophantine Geometry



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# Some Standard Notation

*The following notation is used in a standard way throughout.*

*Rings are assumed commutative and without divisors of 0, unless otherwise specified.*

$\mu$	group of all roots of unity.
$\mu(K)$	subgroup of roots of unity in a field $K$ .
$R^*, K^*$	invertible elements in a ring $R$ (resp. in a field $K$ ).
$K^a$	algebraic closure of a field $K$ .
$K(P)$	field obtained by adjoining to $K$ a set of affine coordinates for a point $P$ (equal to $K(x_0/x_i, \dots, x_n/x_i)$ if $(x_0, \dots, x_n)$ are projective coordinates for $P$ ).
$R(\mathfrak{a})$	$R/\mathfrak{a}$ for any ideal $\mathfrak{a}$ .
$\mathbf{Z}(N)$	$\mathbf{Z}/N\mathbf{Z}$ .
$A^{(p)}$	$p$ -primary part of an abelian group $A$ (that is, the subgroup of elements whose order is a power of $p$ ).
$A_m$	subgroup of elements $x$ in an abelian group $A$ such that $mx = 0$ .
$[n]$	multiplication by an integer $n$ on an abelian group.
$V(K)$	set of $K$ -valued points of a variety or scheme $V$ .
$D \sim D'$	for divisors $D, D'$ , linear equivalence.
$D \approx D'$	for divisors $D, D'$ , algebraic equivalence.
$\text{Div}(V)$	group of divisors on a variety $V$ .
$\text{Div}_a(V)$	subgroup of divisors algebraically equivalent to 0.
$\text{Div}_l(V)$	subgroup of divisors linearly equivalent to 0.



$\text{Pic}(V)$	$\text{Div}(V)/\text{Div}_l(V)$ .
$\text{Pic}_0(V)$	$\text{Div}_a(V)/\text{Div}_l(V)$ .
$\text{NS}(V)$	$\text{Div}(V)/\text{Div}_a(V)$ (the Néron–Severi group of $V$ ).
$h \sim h'$	equivalence for functions, $ h - h' $ is bounded.
$h \approx h'$	quasi-equivalence for functions: for each $\varepsilon > 0$ , $-C_1 + (1 - \varepsilon)h \leq h' \leq (1 + \varepsilon)h + C_2.$
$h \ll h'$	for functions, with $h'$ positive, there exists a constant $C > 0$ such that $ h  \leq Ch'$ . Same as $h = O(h')$ .
$h \gg \ll h'$	both $h, h'$ positive, $h \ll h'$ and $h' \ll h$ .

Usually  $h, H$  denote **heights** with  $h = \log H$ . These are indexed to specify qualifications:

$h_\varphi$	height determined by a morphism $\varphi$ into projective space.
$k_K$	height relative to a field $K$ .
$h_X$	height determined by a morphism derived from the linear system $\mathcal{L}(X)$ , well defined up to $O(1)$ .
$h_c$	on an arbitrary variety, the height associated with a divisor class $c$ , determined only up to $O(1)$ ; on an abelian variety, the canonical height.
$\hat{h}_c$	canonical height if one needs to distinguish it from an equivalence class of heights.

*Standard references:*

IAG = *Introduction to Algebraic Geometry* [L 2].

AV = *Abelian Varieties* [L 3].

Weil's *Foundations* is still quoted in canonical style,  $F^2-X_y$ , Theorem Z, which refers to the second edition.

For schemes, see Hartshorne's *Algebraic Geometry*, and also Mumford's *Abelian Varieties*.