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Readings in Mathematics

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Graduate Texts in Mathematics

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William Fulton Joe Harris

Representation Theory

A First Course

With 144 Illustrations

 Springer

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Preface

The primary goal of these lectures is to introduce a beginner to the finite-dimensional representations of Lie groups and Lie algebras. Since this goal is shared by quite a few other books, we should explain in this Preface how our approach differs, although the potential reader can probably see this better by a quick browse through the book.

Representation theory is simple to define: it is the study of the ways in which a given group may act on vector spaces. It is almost certainly unique, however, among such clearly delineated subjects, in the breadth of its interest to mathematicians. This is not surprising: group actions are ubiquitous in 20th century mathematics, and where the object on which a group acts is not a vector space, we have learned to replace it by one that is (e.g., a cohomology group, tangent space, etc.). As a consequence, many mathematicians other than specialists in the field (or even those who think they might want to be) come in contact with the subject in various ways. It is for such people that this text is designed. To put it another way, we intend this as a book for beginners to learn from and not as a reference.

This idea essentially determines the choice of material covered here. As simple as is the definition of representation theory given above, it fragments considerably when we try to get more specific. For a start, what kind of group G are we dealing with—a finite group like the symmetric group \mathfrak{S}_n or the general linear group over a finite field $GL_n(\mathbb{F}_q)$, an infinite discrete group like $SL_n(\mathbb{Z})$, a Lie group like $SL_n\mathbb{C}$, or possibly a Lie group over a local field? Needless to say, each of these settings requires a substantially different approach to its representation theory. Likewise, what sort of vector space is G acting on: is it over \mathbb{C} , \mathbb{R} , \mathbb{Q} , or possibly a field of positive characteristic? Is it finite dimensional or infinite dimensional, and if the latter, what additional structure (such as norm, or inner product) does it carry? Various combinations

of answers to these questions lead to areas of intense research activity in representation theory, and it is natural for a text intended to prepare students for a career in the subject to lead up to one or more of these areas. As a corollary, such a book tends to get through the elementary material as quickly as possible: if one has a semester to get up to and through Harish–Chandra modules, there is little time to dawdle over the representations of \mathfrak{S}_4 and $\mathrm{SL}_3\mathbb{C}$.

By contrast, the present book focuses exactly on the simplest cases: representations of finite groups and Lie groups on finite-dimensional real and complex vector spaces. This is in some sense the common ground of the subject, the area that is the object of most of the interest in representation theory coming from outside.

The intent of this book to serve nonspecialists likewise dictates to some degree our approach to the material we do cover. Probably the main feature of our presentation is that we concentrate on examples, developing the general theory sparingly, and then mainly as a useful and unifying language to describe phenomena already encountered in concrete cases. By the same token, we for the most part introduce theoretical notions when and where they are useful for analyzing concrete situations, postponing as long as possible those notions that are used mainly for proving general theorems.

Finally, our goal of making the book accessible to outsiders accounts in part for the style of the writing. These lectures have grown from courses of the second author in 1984 and 1987, and we have attempted to keep the informal style of these lectures. Thus there is almost no attempt at efficiency: where it seems to make sense from a didactic point of view, we work out many special cases of an idea by hand before proving the general case; and we cheerfully give several proofs of one fact if we think they are illuminating. Similarly, while it is common to develop the whole semisimple story from one point of view, say that of compact groups, or Lie algebras, or algebraic groups, we have avoided this, as efficient as it may be.

It is of course not a strikingly original notion that beginners can best learn about a subject by working through examples, with general machinery only introduced slowly and as the need arises, but it seems particularly appropriate here. In most subjects such an approach means one has a few out of an unknown infinity of examples which are useful to illuminate the general situation. When the subject is the representation theory of complex semisimple Lie groups and algebras, however, something special happens: once one has worked through all the examples readily at hand—the “classical” cases of the special linear, orthogonal, and symplectic groups—one has not just a few useful examples, one has all but five “exceptional” cases.

This is essentially what we do here. We start with a quick tour through representation theory of finite groups, with emphasis determined by what is useful for Lie groups. In this regard, we include more on the symmetric groups than is usual. Then we turn to Lie groups and Lie algebras. After some preliminaries and a look at low-dimensional examples, and one lecture with

some general notions about semisimplicity, we get to the heart of the course: working out the finite-dimensional representations of the classical groups.

For each series of classical Lie algebras we prove the fundamental existence theorem for representations of given highest weight by explicit construction. Our object, however, is not just existence, but to see the representations in action, to see geometric implications of decompositions of naturally occurring representations, and to see the relations among them caused by coincidences between the Lie algebras.

The goal of the last six lectures is to make a bridge between the example-oriented approach of the earlier parts and the general theory. Here we make an attempt to interpret what has gone before in abstract terms, trying to make connections with modern terminology. We develop the general theory enough to see that we have studied all the simple complex Lie algebras with five exceptions. Since these are encountered less frequently than the classical series, it is probably not reasonable in a first course to work out their representations as explicitly, although we do carry this out for one of them. We also prove the general Weyl character formula, which can be used to verify and extend many of the results we worked out by hand earlier in the book.

Of course, the point we reach hardly touches the current state of affairs in Lie theory, but we hope it is enough to keep the reader's eyes from glazing over when confronted with a lecture that begins: "Let G be a semisimple Lie group, P a parabolic subgroup, ..." We might also hope that working through this book would prepare some readers to appreciate the elegance (and efficiency) of the abstract approach.

In spirit this book is probably closer to Weyl's classic [We1] than to others written today. Indeed, a secondary goal of our book is to present many of the results of Weyl and his predecessors in a form more accessible to modern readers. In particular, we include Weyl's constructions of the representations of the general and special linear groups by using Young's symmetrizers; and we invoke a little invariant theory to do the corresponding result for the orthogonal and symplectic groups. We also include Weyl's formulas for the characters of these representations in terms of the elementary characters of symmetric powers of the standard representations. (Interestingly, Weyl only gave the corresponding formulas in terms of the exterior powers for the general linear group. The corresponding formulas for the orthogonal and symplectic groups were only given recently by D'Hoker, and by Koike and Terada. We include a simple new proof of these determinantal formulas.)

More about individual sections can be found in the introductions to other parts of the book.

Needless to say, a price is paid for the inefficiency and restricted focus of these notes. The most obvious is a lot of omitted material: for example, we include little on the basic topological, differentiable, or analytic properties of Lie groups, as this plays a small role in our story and is well covered in dozens of other sources, including many graduate texts on manifolds. Moreover, there are no infinite-dimensional representations, no Harish-Chandra or Verma

modules, no Stiefel diagrams, no Lie algebra cohomology, no analysis on symmetric spaces or groups, no arithmetic groups or automorphic forms, and nothing about representations in characteristic $p > 0$. There is no consistent attempt to indicate which of our results on Lie groups apply more generally to algebraic groups over fields other than \mathbb{R} or \mathbb{C} (e.g., local fields). And there is only passing mention of other standard topics, such as universal enveloping algebras or Bruhat decompositions, which have become standard tools of representation theory. (Experts who saw drafts of this book agreed that some topic we omitted must not be left out of a modern book on representation theory—but no two experts suggested the same topic.)

We have not tried to trace the history of the subjects treated, or assign credit, or to attribute ideas to original sources—this is far beyond our knowledge. When we give references, we have simply tried to send the reader to sources that are as readable as possible for one knowing what is written here. A good systematic reference for the finite-group material, including proofs of the results we leave out, is Serre [Se2]. For Lie groups and Lie algebras, Serre [Se3], Adams [Ad], Humphreys [Hu1], and Bourbaki [Bour] are recommended references, as are the classics Weyl [We1] and Littlewood [Lit1].

We would like to thank the many people who have contributed ideas and suggestions for this manuscript, among them J-F. Burnol, R. Bryant, J. Carrell, B. Conrad, P. Diaconis, D. Eisenbud, D. Goldstein, M. Green, P. Griffiths, B. Gross, M. Hildebrand, R. Howe, H. Kraft, A. Landman, B. Mazur, N. Chriss, D. Petersen, G. Schwartz, J. Towber, and L. Tu. In particular, we would like to thank David Mumford, from whom we learned much of what we know about the subject, and whose ideas are very much in evidence in this book.

Had this book been written 10 years ago, we would at this point thank the people who typed it. That being no longer applicable, perhaps we should thank instead the National Science Foundation, the University of Chicago, and Harvard University for generously providing the various Macintoshes on which this manuscript was produced. Finally, we thank Chan Fulton for making the drawings.

Bill Fulton and Joe Harris

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Using This Book

A few words are in order about the practical use of this book. To begin with, prerequisites are minimal: we assume only a basic knowledge of standard first-year graduate material in algebra and topology, including basic notions about manifolds. A good undergraduate background should be more than enough for most of the text; some examples and exercises, and some of the discussion in Part IV may refer to more advanced topics, but these can readily be skipped. Probably the main practical requirement is a good working knowledge of multilinear algebra, including tensor, exterior, and symmetric products of finite dimensional vector spaces, for which Appendix B may help. We have indicated, in introductory remarks to each lecture, when any background beyond this is assumed and how essential it is.

For a course, this book could be used in two ways. First, there are a number of topics that are not logically essential to the rest of the book and that can be skimmed or skipped entirely. For example, in a minimal reading one could skip §§4, 5, 6, 11.3, 13.4, 15.3–15.5, 17.3, 19.5, 20, 22.1, 22.3, 23.3–23.4, 25.3, and 26.2; this might be suitable for a basic one-semester course. On the other hand, in a year-long course it should be possible to work through as much of the material as background and/or interest suggested. Most of the material in the Appendices is relevant only to such a long course. Again, we have tried to indicate, in the introductory remarks in each lecture, which topics are inessential and may be omitted.

Another aspect of the book that readers may want to approach in different ways is the profusion of examples. These are put in largely for didactic reasons: we feel that this is the sort of material that can best be understood by gaining some direct hands-on experience with the objects involved. For the most part, however, they do not actually develop new ideas; the reader whose tastes run more to the abstract and general than the concrete and special may skip many

of them without logical consequence. (Of course, such a reader will probably wind up burning this book anyway.)

We include hundreds of exercises, of wildly different purposes and difficulties. Some are the usual sorts of variations of the examples in the text or are straightforward verifications of facts needed; a student will probably want to attempt most of these. Sometimes an exercise is inserted whose solution is a special case of something we do in the text later, if we think working on it will be useful motivation (again, there is no attempt at “efficiency,” and readers are encouraged to go back to old exercises from time to time). Many exercises are included that indicate some further directions or new topics (or standard topics we have omitted); a beginner may best be advised to skim these for general information, perhaps working out a few simple cases. In exercises, we tried to include topics that may be hard for nonexperts to extract from the literature, especially the older literature. In general, much of the theory is in the exercises—and most of the examples in the text.

We have resisted the idea of grading the exercises by (expected) difficulty, although a “problem” is probably harder than an “exercise.” Many exercises are starred: the * is not an indication of difficulty, but means that the reader can find some information about it in the section “Hints, Answers, and References” at the back of the book. This may be a hint, a statement of the answer, a complete solution, a reference to where more can be found, or a combination of any of these. We hope these miscellaneous remarks, as haphazard and uneven as they are, will be of some use.

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