

N. Pytheas Fogg

# Substitutions in Dynamics, Arithmetics and Combinatorics

Editors: V. Berthé  
S. Ferenczi  
C. Mauduit  
A. Siegel



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## Author

N. Pytheas Fogg  
Marseille, France

## Editors

Valérie Berthé  
Sébastien Ferenczi  
Christian Mauduit  
Univ. de la Méditerranée  
IML, Case 907  
163 av. de Luminy  
13288 Marseille Cedex 09, France

Anne Siegel  
IRISA  
Campus de Beaulieu  
35042 Rennes Cedex  
France  
*e-mail: Anne.Siegel@irisa.fr*

*e-mail:*  
*berthe@iml.univ-mrs.fr*  
*ferenczi@iml.univ-mrs.fr*  
*mauduit@iml.univ-mrs.fr*

Authors' name: N. stands for "no nomen". Pytheas was the Greek navigator and scientist (4th century B.C.) who became one of the historical heroes of the city of Marseille (Massalia). Phileas Fogg was the excentric and phlegmatic hero of Jules Verne who travelled around the world by "jumping mathematically from train to boat, never forgetting his sense of humour."

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