

# Springer Series in Computational Physics

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**Editors:** J.-L. Armand M. Holt P. Hut  
H. B. Keller J. Killeen S. A. Orszag V. V. Rusanov

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C.A.J. Fletcher

# Computational Techniques for Fluid Dynamics 1

Fundamental and General Techniques

With 138 Figures



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**Dr. Clive A. J. Fletcher**

Department of Mechanical Engineering, The University of Sydney  
New South Wales 2006, Australia

*Editors*

**J.-L. Armand**

Department of Mechanical Engineering  
University of California  
Santa Barbara, CA 93106, USA

**M. Holt**

College of Engineering and  
Mechanical Engineering  
University of California  
Berkeley, CA 94720, USA

**P. Hut**

The Institute for Advanced Study  
School of Natural Sciences  
Princeton, NJ 08540, USA

**H. B. Keller**

Applied Mathematics 101-50  
Firestone Laboratory  
California Institute of Technology  
Pasadena, CA 91125, USA

**J. Killeen**

Lawrence Livermore Laboratory  
P.O. Box 808  
Livermore, CA 94551, USA

**S. A. Orszag**

Applied and Computational Mathematics,  
218 Fine Hall, Princeton University,  
Princeton, NJ 08544, USA

**V. V. Rusanov**

Keldysh Institute of Applied Mathematics  
4 Miusskaya Pl.  
SU-125047 Moscow, USSR

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## Preface

The purpose of this two-volume textbook is to provide students of engineering, science and applied mathematics with the specific techniques, and the framework to develop skill in using them, that have proven effective in the various branches of computational fluid dynamics (CFD). Volume 1 describes both fundamental and general techniques that are relevant to all branches of fluid flow. Volume 2 provides specific techniques, applicable to the different categories of engineering flow behaviour, many of which are also appropriate to convective heat transfer.

An underlying theme of the text is that the competing formulations which are suitable for computational fluid dynamics, e.g. the finite difference, finite element, finite volume and spectral methods, are closely related and can be interpreted as part of a unified structure. Classroom experience indicates that this approach assists, considerably, the student in acquiring a deeper understanding of the strengths and weaknesses of the alternative computational methods.

Through the provision of 24 computer programs and associated examples and problems, the present text is also suitable for established research workers and practitioners who wish to acquire computational skills without the benefit of formal instruction. The text includes the most up-to-date techniques and is supported by more than 300 figures and 500 references.

For the conventional student the contents of Vol. 1 are suitable for introductory CFD courses at the final-year undergraduate or beginning graduate level. The contents of Vol. 2 are applicable to specialised graduate courses in the engineering CFD area. For the established research worker and practitioner it is recommended that Vol. 1 is read and the problems systematically solved before the individual's CFD project is started, if possible. The contents of Vol. 2 are of greater value after the individual has gained some CFD experience with his own project.

It is assumed that the reader is familiar with basic computational processes such as the solution of systems of linear algebraic equations, non-linear equations and ordinary differential equations. Such material is provided by Dahlquist, Bjorck and Anderson in *Numerical Methods*; by Forsythe, Malcolm and Moler in *Computer Methods for Mathematical Computation*; and by Carnaghan, Luther and Wilkes in *Applied Numerical Analysis*. It is also assumed that the reader has some knowledge of fluid dynamics. Such knowledge can be obtained from *Fluid Mechanics* by Streeter and

Wylie; from *An Introduction to Fluid Dynamics* by Batchelor; or from *Incompressible Flow* by Panton, amongst others.

Computer programs are provided in the present text for guidance and to make it easier for the reader to write his own programs, either by using equivalent constructions, or by modifying the programs provided. In the sense that the CFD practitioner is as likely to inherit an existing code as to write his own from scratch, some practice in modifying existing, but simple, programs is desirable. An IBM-compatible floppy disk containing the computer programs may be obtained from the author.

The contents of Vol. 1 are arranged in the following way; Chapter 1 contains an introduction to computational fluid dynamics, designed to give the reader an appreciation of why CFD is so important, the sort of problems it is capable of solving and an overview of how CFD is implemented. The equations governing fluid flow are usually expressed as partial differential equations. Chapter 2 describes the different classes of partial differential equations, appropriate boundary conditions and briefly reviews traditional methods of solution.

Obtaining computational solutions consists of two stages, the reduction of the partial differential equations to algebraic equations and the solution of the algebraic equations. The first stage, called discretisation, is examined in Chap. 3 with special emphasis on the accuracy. Chapter 4 provides sufficient theoretical background to ensure that computational solutions can be related properly to the usually unknown “exact” solution. Weighted residual methods are introduced in Chap. 5 as a vehicle for investigating and comparing the finite element, finite volume and spectral methods as alternative means of discretisation. Specific technique to solve the algebraic equations resulting from discretisation are described in Chap. 6. Chapters 3–6 provide essential background information.

The one-dimensional diffusion equation, considered in Chap. 7, provides the simplest model for highly dissipative fluid flows. This equation is used to contrast explicit and implicit methods and to discuss the computational representation of derivative boundary conditions. If two or more spatial dimensions are present, splitting techniques are usually required to obtain computational solutions efficiently. Splitting techniques are described in Chap. 8. Convective (or advective) aspects of fluid flow, and their effective computational prediction, are examined in Chap. 9. The convective terms are usually nonlinear. The additional difficulties that this introduces are considered in Chap. 10. The general techniques, developed in Chaps. 7–10, are utilised in constructing specific techniques for the different categories of flow behaviour, as is demonstrated in Chaps. 14–18 of Vol. 2.

In preparing this textbook I have been assisted by many people. In particular I would like to thank Dr. K. Srinivas, Nam-Hyo Cho and Zili Zhu for having read the text and made many helpful suggestions. I am grateful

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Sydney, October 1987

*C. A. J. Fletcher*

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