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Robert Finn

Equilibrium Capillary Surfaces

With 98 Illustrations

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To the fantasy and creativity of youth,
that they serve to build and not to destroy

Preface

Capillarity phenomena are all about us; anyone who has seen a drop of dew on a plant leaf or the spray from a waterfall has observed them. Apart from their frequently remarked poetic qualities, phenomena of this sort are so familiar as to escape special notice. In this sense the rise of liquid in a narrow tube is a more dramatic event that demands and at first defied explanation; recorded observations of this and similar occurrences can be traced back to times of antiquity, and for lack of explanation came to be described by words deriving from the Latin word "capillus", meaning hair.

It was not until the eighteenth century that an awareness developed that these and many other phenomena are all manifestations of something that happens whenever two different materials are situated adjacent to each other and do not mix. If one (at least) of the materials is a fluid, which forms with another fluid (or gas) a free surface interface, then the interface will be referred to as a *capillary surface.*

Attempts to explain observed phenomena go back at least to Leonardo da Vinci. A consistent theory capable of scientific prediction first appears however in the writings of Young and of Laplace in the early nineteenth century. The theory was later put onto a more solid foundation by Gauss, and it became the object of extensive study by some of the most imposing scientific figures of that century (although it must be remarked that very little more of major new interest was accomplished). The problem fell out of fashion during the first half of the present century; however, the impetus on the one hand of new mathematical developments on minimal surfaces, and on the other hand of the practical demands of space age technology and of medicine, have now led to renewed activity on several fronts.

Among mathematical developments, the BV theory, founded on the ideas of Caccioppoli and of de Giorgi and developed by Miranda, Giaquinta, Anzellotti, Massari, Tamanini, and others, led to the first general existence theorem for capillary surfaces (Emmer [46]). Independently the ideas of geometric measure theory were introduced and developed by Federer, Fleming, Almgren, Allard, and others, and were used effectively by Taylor [177] to prove boundary regularity.

From an engineering point of view, specific problems have been attacked energetically using traditional methods, chiefly that of matching expansions (due originally, incidentally, to Laplace), and also numerically with computers. In general, good results were obtained; however, in some particular situations the procedures led unexpectedly to incoherent answers.

It is this circumstance that attracted my own interest. A direct study of the underlying equations showed that a discontinuous dependence on data occurs, which is governed by the particular nonlinearity in the equations. Unconventional but simple procedures led to a precise characterization of the criterion for singular behavior, to general bounds on solutions, and to asymptotically exact information in some cases.

It turned out that various other problems also lent themselves to analogous (phenomenological) approaches. By now a number of studies have appeared by various authors, using varying methods and occasionally with striking conclusions. A common and unifying thread is appearing, which may not be evident on reading the individual papers. I hope the detailed results presented in the following chapters will be of interest in themselves, and that their juxtaposition under a single cover will help to bring the thread into visibility.

The exposition is not intended to be encyclopedic, and the omission of a particular result in no sense implies that I regard it less highly than material I have included. I have tried to illustrate by example the varying kinds of situations that may be encountered, and in each case the choice of example has been determined largely by the simple criterion of familiarity. Thus, the work of Vogel [182, 183, 184] on liquid bridges, and of Turkington [180] on exterior problems and extension to a class of nonlinear operators, each of which I hold in high regard, is omitted by circumstance and not by design.

A glance through the Contents should indicate the specific nature of the material that has been covered. Much of it refers to particular configurations that may be taken as cases of special interest in the context of the general existence (and nonexistence) results of Chapters 6 and 7. Also for these general results the exposition is not complete, my intention being to emphasize the underlying ideas and the unifying thread. Attention is directed throughout to the unexpected, in the sense of behavior that differs qualitatively from what would be predicted by usual perturbation or linearizing procedures.

In the interest of conceptual and notational simplicity all material in the text is presented for the (physical) case of two-dimensional surfaces in 3-space. Many – but not all – of the results extend without essential change to surfaces of codimension one in n-space.

The purview of this book is limited to equilibrium configurations. It is not limited to energy minimizing configurations. The equations are not cognizant of global energy relations, and can lead to interesting solutions that are not observed physically as a global entity. Some of these are studied in Chapter 4.

Time dependent situations present a different world that will require a different book, presumably by a different author. We mention however recent work by Bemelmans [9], Pukhnachev and Solonnikov [151], and Dussan V and Chow [44].

I am indebted to many colleagues and students for comments and discussions that have done much to clarify my understanding and to shape my point of view. The book has also profited immeasurably from my long collaboration with Paul Concus. As to the specific material in the text, J.B. Keller has made helpful comments with regard to Chapter 1. Chapters 6 and 7 have benefited greatly from observations by L.F. Tam. I wish especially to thank Enrico Giusti, who generously shared with me his deep insight during the course of many conversations over many years. Giusti also read Chapters 6 and 7 in detail, and his comments led to a number of improvements in the formulations and proofs of the results.

Much of the writing was done while I was visiting at Universität Bonn under the auspices of Sonderforschungsbereich 72. I owe a special debt of gratitude to Stefan Hildebrandt for his warm hospitality and for the stimulating conditions for scientific work in the Institute he directs.

The new research presented here was supported in part by the National Science Foundation and by the National Aeronautics and Space Administration.

The larger portion of the typing was done by Charlotte Crabtree, who also prepared most of the figures. I want to thank her not only for her elegant work, but also for her patience with me in the course of many rewritings and changes. I also wish to thank Anke Vogt for her excellent typing of the remainder of the material.

This book has gained in accuracy and readability from the scrupulous attention its production editor gave to layout and detail. My thanks are due also to the compositor for precise and careful work.

Finally I want to express my appreciation to Springer-Verlag for its patience and understanding while awaiting a long overdue manuscript, and for its generous attention to details of production.

Palo Alto, California
November, 1985

ROBERT FINN

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IX. *Part of a Letter from Mr.* Brook Taylor, *F. R. S. to Dr.* Hans Sloane *R. S. Secr. Concerning the Ascent of Water between two Glass Planes.*

THE following Experiment seeming to be of use, in discovering the Proportions of the Attractions of Fluids, I shall not forbear giving an Account of it; tho' I have not here Conveniencies to make it in so successful a manner, as I could wish.

I fasten'd two pieces of Glass together, as flat as I could get; so that they were inclined in an Angle of about 2 Degrees and a half. Then I set them in Water, with the contiguous Edges perpendicular. The upper part of the Water, by rising between them, made this *Hyperbola*; [See *Fig.* 5.] which is as I copied it from the Glass.

I have examined it as well as I can, and it seems to approach very near to the common *Hyperbola*. But my *Apparatus* was not nice enough to discover this exactly.

The Perpendicular *Assymptote* was exactly determined by the Edge of the Glass; but the Horizontal one I could not so well discover. I am,

Sir,

Bifrons near *Canterbury*, June 25. 1712

Your most humble Servant,

BROOK TAYLOR.

X. *An Account of an Experiment touching the Ascent of Water between two Glass Planes, in an Hyperbolick Figure. By Mr.* Francis Hauksbee, *F. R. S.*

I Took two Glass Planes, each somewhat more than 20 Inches long, of the truest Surfaces I could procure. These being held close together at one of their Ends, the other Ends were opened exactly to an Angle of 20 Minutes. In this Form they were edgeways put

into a Trough of ting'd Water, which immediately aroſe between them in the Figure of the annext Scheme. *See* Fig. 7.
At another time the Planes were opened to an Angle of 40 Minutes; then the Water appear'd between them, as in the Scheme with that Title. By theſe Schemes *See* Fig. 6.
the Proportions of the Power of Attraction are in ſome meaſure evident to the Eye; for there may be ſeen at the ſeveral Diſtances, how many Lines (which are 12ths of Inches) the Water is elevated, and the prodigious Increaſe of them near the touching Ends. I hope the Tables are pretty accurate; for after many tryals, I find the Succeſſes to be much the ſame, according to the different Angles. This Experiment was firſt made by Mr. *Brook Taylor*, as appears by his Letter to Dr. *Hans Sloane*, R. S. Secr. but he confeſſes his *Apparatus* not nice enough to diſcover exactly the Figure which the Water made between the Planes.