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Algebra II
Ring Theory

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This volume is for my family

Mickey, Heidi, and Cindy
Eldridge, Louise, and Frederick
Virginia Nell Caudill Compton
Harold Compton 1895–1964

and the memory of my parents

Herbert Spencer Faith 1895–1952
Vila Belle Foster Faith 1897–1965

Preface to Volume II

I. Ring Theory

The term *The Theory of Rings* seems first used as title of a book by Jacobson [43], and in his preface Jacobson asserts that the theory that forms the subject of the book had its beginning with Artin's extension in 1927 of Wedderburn's structure theory of algebras to rings satisfying the chain conditions.[1]

As the predecessor to his book, Jacobson cites Deuring's *Algebren* (Deuring [35, 68]), and Deuring cites Dickson's *Algebren und ihre Zahlentheorie* (Zürich, 1927). As in his earlier book, *Algebra and its Arithmetic*, Dickson ([23]) extends arithmetic in algebraic number field, that is, arithmetic of the ring of integers in a finite field extension k of the field $\mathbb{Q}$ of rational numbers, to orders in rational algebras, that is, to orders in an algebra over k.

Jacobson also cites nine papers of his teacher, J.H.M. Wedderburn, in the bibliography; the term "algebra" appears in six titles, and "hypercomplex numbers" in another. (Another influential book, appearing shortly before Jacobson's, by A.A. Albert in 1939, was also on the subject of "algebras", that is, algebras with a finite basis over a field.) The study of these so called "hypercomplex number" systems was motivated originally by the desire to discover and classify algebras over the field of real or complex numbers (thus the terminology: hypercomplex).

Hamilton's discovery of the algebra $\mathbb{H}$ of quaternions, the first noncommutative field, motivated by a problem in physics, took him 15 years to find. (There is a legend that he carved the result on a nearby bridge the moment of discovery—see E.T. Bell [37, p. 360].) Another discovery of importance to physics, the Cayley numbers were a nonassociative field (containing $\mathbb{H}$) of dimension 8 over $\mathbb{R}$. (Unlike Hamilton with his Quaternions, Cayley did not write a treatise on the subject purporting to explain the physical universe. Cf. Dyson [72, p. 301, Note 5].)

Wedderburn's theorems apply to the structure of any algebra A of finite dimension over any field k: if $W(A)$ is the maximal nilpotent ideal, then $A/W(A)$ is a finite product of algebras each isomorphic to a total matrix algebra over a (noncommutative or commutative) field. If A is a separable algebra (the case when the center of $A/W(A)$, a finite product of fields, has the property that each of the fields is a separable extension of k), then there is a subalgebra $S \approx A/W(A)$

[1] Our convention was stated in Volume I: Jacobson [43], for example, denotes a work by Jacobson published in 1943. When more than one work appears, small letters are used as in Jacobson [45a], [45b], or [45c].

such that

$$A = S \oplus W(A)$$

as vector spaces over k (Wedderburn factor theorem (I,13.18, p. 471)).[2] This reduces the structure theory of A to that of S and $W(A)$ and the effect of the multiplication of $W(A)$ by the elements of S. (The latter effect is mostly conclusive only in determining the structure of A only in low dimensions, however.)

In 1929 R. Brauer showed that the "classes" of simple central algebras over k formed a group $Br(k)$. For each such algebra A, there is a class $[A]$ consisting of all algebras B for which there exist integers m and n such that A_n and B_m are isomorphic total matrix algebras of degrees n and m over A and B respectively. (By Wedderburn's theorem, $[A]$ contains a (not necessarily commutative) field D over k, that is $[A]=[D]$.) In $Br(k)$ we have $[A]^{-1}=[A^{op}]$, where A^{op} is the algebra opposite to A, and, moreover, $Br(k)$ is a torsion abelian group. (In Chapter 13 Exercises, the Brauer group $Br(k)$ of any commutative ring k is discussed.)

In 1921, E. Noether carried over Dedekind's ideal theory (and representation theory) for integral domains (and rings of algebraic integers) to general commutative rings satisfying the ascending chain condition for ideals. Such rings are now called Noetherian rings. These rings are characterized by the condition that every ideal is finitely generated, and they include polynomial rings in any finite number of variables over any field (Hilbert's basis theorem (I, 7.13, p. 341)) in addition to Dedekind rings, and other rings arising in classical mathematics. Moreover, the study of modules over these rings formed an important part of the arithmetic ideal theory (see Noether [21, p. 55ff]).

In 1927, Noether proved the "Noether homomorphism theorems" for groups with operators, generalizing to modules many of the group theoretical theorems of W. Krull and O. Schmidt (see Noether [27, p. 643 and 645]).

In 1927, Artin generalized some of the Wedderburn theorems for algebras to noncommutative (= not necessarily commutative) rings satisfying the descending chain for right ideals.[3] These have been called right Artinian rings in his honor. This freed the subject from its earlier dependence on an underlying field k of scalars and finite free basis over k. The ascending chain condition for right ideals was required in Artin's rings a restriction which was shown to be superfluous[4] only much later, independently by C. Hopkins [39] and J. Levitzki [39]. (An account of this appears in Chapter 18.)

From all indications, the writing of "Theory of Rings" stimulated Jacobson researches in ring theory to fruition in a series of historically most important papers (Jacobson [45a, b, c]). In essence, for the first time, the full category RINGS was studied in order to more fully understand fundamental concepts and not just because important subcategories arise in classical mathematics: in particular, no chain conditions, ascending or otherwise, were assumed (although

[2] For reference to Volume I: (I, 13.18, p. 471) denotes an item (Theorem, Proposition, Exercise, or Corollary) on p. 471. Also: 13.18 (I, p. 471) is a variant reference for this.

[3] Bourbaki [58, Chapter 8, p. 174] gives the interesting history of the notions of minimal ideal of an algebra (H. Poincarë, 1903), one-sided ideals (Noether and Schmeidler, 1920), the maximal or ascending chain condition (Dedekind, 1894), and the descending chain condition (Wedderburn, 1907).

[4] Bourbaki (l.c., p. 175) remarks that Noether [29] dispensed with the ascending chain condition assuming no nilpotent ideals.

applications were given) and, not even an identity element was required. (And, of course, commutativity was not assumed.)

No doubt the single most important of these ideas is that of the Jacobson radical rad R of a ring R, defined as the intersection of all kernels of all (nontrivial) irreducible right representations of R, and characterized by Jacobson as the unique maximal ideal J with the property that

(q.r.) $\qquad \forall_{x \in J} \exists_{x' \in R}\, x + x' + x x' = x + x' + x' x = 0$

and containing any one-sided ideal for which the same q.r. condition holds. In characterizing the radical this way, Jacobson seized on the characterization of Perlis [42] of the radical of a finite dimensional algebra over a field, and extended it to an arbitrary ring. (The term quasi-regular element x (= one satisfying the q.r. condition just defined) was coined by Perlis, and a quasi-regular ideal is one in which every element is quasi-regular.)

In other words, Jacobson defined a functor

$$\text{rad}: \text{RINGS} \rightsquigarrow \text{RINGS}$$

and we now list some of its properties.

1. For a ring R, rad R has been defined via irreducible right modules (or representations) of R, and thus should be called the "right" radical of R, but Jacobson proved that rad R coincides with its left-right symmetry, that is, rad R is also the left radical of R. Moreover:

$$\text{rad}\,(R/\text{rad}\,R) = 0.$$

2. The characterization of rad R as the intersection of all right ideals I such that the right module $V_I = R/I$ is irreducible (= simple) and nontrivial ($V_I R \neq 0$). If R has an identity element, then any such right ideal is a maximal right ideal and conversely. (Then rad R is the intersection of the maximal left ideals.) The existence of these in any ring R with identity element 1 follows by an application of Zorn's lemma; of course, a nilpotent ring does not possess nontrivial simple modules. (Incidentally, in case $1 \in R$, then $x \in R$ is quasi-regular iff $1 + x$ is a unit.)

Before proceeding, we need a definition. An ideal I is right primitive if R/I has a faithful irreducible right module. A ring R is primitive if 0 is a right primitive ideal.

By definition, rad R is the intersection of the right primitive ideals, and by the result in 1., rad R is the intersection of the left primitive ideals. Thus, a simple ring is both right and left primitive.

3. A ring R is right primitive iff R is isomorphic to a dense ring of linear transformations on left vector space V over a field D (Chevalley-Jacobson Density Theorem, proved in Jacobson [45]).

If R is right primitive, with faithful irreducible right module V, then $D = \text{End}\, V_R$ is a field, and (by writing endomorphisms on the left) V becomes a left vector space over D. Moreover, R imbeds in $L = \text{End}_D V$ canonically, and is dense in L in the finite topology.

Any ring with 1 has right primitive ideals. Any maximal ideal is primitive, since any simple ring (with 1) is right and left primitive (Jacobson [45b]). Thus, any ring $R \neq \text{rad}\, R$ is "interesting" in that it has a "good" ring, namely a primitive ring, as an epic image.[5] (Doubtlessly, it is this theorem which establishes the importance of RINGS as a category.)

4. The factor ring $R/\text{rad}\,R$ is either $=0$, or isomorphic to a subdirect product of right primitive rings. (Also the left-right symmetry holds even though not every right primitive ring is left primitive as G. Bergman [64] showed.) Conversely, any subdirect product A of right primitive rings satisfies $\text{rad}\,A=0$.

5. $\text{rad}\,R$ is "functorial" in the sense that any category equivalence

$$T\colon \text{mod-}R \rightsquigarrow \text{mod-}S$$

for rings R and S induces a category equivalence

$$\text{mod-}(R/\text{rad}\,R) \rightsquigarrow \text{mod-}(S/\text{rad}\,S)$$

(Jacobson proved this in different language).

6. $\text{rad}\,R$ contains every nil one-sided ideal (= one in which every element x is nilpotent in the sense that $x^n=0$ for an integer n depending on x), and hence $\text{rad}\,R$ contains every nilpotent one-sided ideal (= one in which there is a fixed integer N such that all products $x_1 \cdot x_2 \cdots x_N$ of N elements $x_1, \ldots, x_N$ vanish: $x_1 x_2 \cdots x_N = 0$).

The Wedderburn radical $W(R)$ of a ring R is the maximal nilpotent ideal (if it exists). In a commutative ring R, the "radical" of an ideal I is the ideal

$$\sqrt{I} = \{x \mid \exists_{n=n(x)} x^n \in I\}.$$

Then $\sqrt{0}$ is called the nil radical of R. (If R is Noetherian, then $\sqrt{I}$ is nilpotent modulo I, and hence in this case $W(R/I)=\sqrt{I}/I$. Thus, $W(R)$ is then the nil radical of R, whence the origin of the generic term radical.) In a noncommutative ring, $W(R)$ exists if R is either right or left Noetherian, and, if R is right Artinian $\text{rad}\,R$ coincides with $W(R)$.

Krull [50] pointed out the relationship between Hilbert's Nullstellensatz and the Jacobson radical. It hinges on the question: when is the Jacobson radical of a finitely generated algebra over a field a nil ideal? This holds true for commutative algebras (Krull [51], Goldman [51]), algebras over nondenumerable fields (Amitsur [56]), and algebras satisfying a polynomial identity (Amitsur [57]). The latter theorem is related to a noncommutative Hilbert Nullstellensatz, and many of the foregoing results on Jacobson and Hilbert rings are generalized by Amitsur and Procesi [66] and Procesi [67].

It is tempting to extend this list of contributions of Jacobson, but, of course, this has been done by him much better in his Colloquium volume (and elsewhere) and much of Volume II involves Jacobson's ring-theoretical ideas in an essential way. In addition, two others who exploited and advanced Jacobson's ringtheoretical techniques, notably in work on Kurosch's problem, rings with polynomial identities, topological rings, Lie and Jordan simplicity of simple associative rings, and the so-called "commutativity theorems" (modelled after the famous Wedderburn Theorem on the commutativity of finite fields), among many, many others,

[5] "Bad" rings are also interesting from different points of view: do there exist simple rings equal to their radicals? (Yes. Cohn and Saciada [67]). Is the radical of a finitely generated ring nil? (See 6.) When does nil $\Rightarrow$ nilpotence? (See Chapter 17.)

have written wisely and well on the subject—I am speaking of I. Kaplansky and I. N. Herstein from whose books I have reaped so much pleasure and knowledge.[6]

II. Module Theory

One might define module theory to be the structure theory of modules satisfying specific conditions, for example, Noetherian, or Artinian, or semiperfect, or indecomposable, without making stringent requirements on the ring. As an example, any Noetherian or Artinian module may be decomposed over *any* ring into a direct sum of indecomposable modules. It would include, of course, the structure of modules over specified rings, for example, Noetherian, or Artinian, or semiperfect, or indecomposable. Another aspect of module theory is the relation between a module and certain canonical modules, such as the right ideals $\{I_a\}_{a\in A}$ of a ring R, and the cyclic modules $\{R/I_a\}_{a\in A}$. These modules we might say are "at hand", and they form a set as opposed to being a class. (Curiously, "right ideals" and "cyclic right modules" are dual in the sense that one is the class of subobjects, and the other the class of quotient objects of R.) Classically, it has been possible to relate much of the class mod-R to this set. (Heuristically, one might compare the problem of doing this with the problem of knowing the universe with only a telescope at hand.)

Nevertheless, even for $R=\mathbb{Z}$, the ring of rational integers, these modules describe all finitely generated abelian groups. A similar theorem holds for finitely generated modules over a hereditary Noetherian prime (HNP) ring R: every finitely generated module M is a direct sum of uniserial (= has a unique decomposition series) modules and right ideals. This follows since Kaplansky's theorem (I, p. 387) for modules over hereditary rings implies that the torsion submodule $t(M)$ splits off, and that $M/t(M)$ is isomorphic to a direct sum of right ideals; then the theorem of Eisenbud, Griffith and Robson (25.5.1) applies: modulo any nonzero ideal R is a (generalized uni) serial ring, so by Nakayama's theorem (25.4), $M\approx t(M)\oplus M/t(M)$ is isomorphic to a direct sum of uniserial modules and right ideals.

A ring need not be right Noetherian in order that the finitely generated modules have such a decomposition. Indeed, a theorem of Kaplansky [52] states that over any almost maximal valuation ring, any finitely generated module is a direct sum of cyclic modules. In fact, this property characterizes almost maximal valuation rings among commutative local rings (20.49).

However, a stronger requirement does imply right Noetherian. Assume that there is a set S of right R-modules such that every right R-module can be embedded in a direct sum of modules in the set. Then, R is right Noetherian (Faith-Walker (20.7)). The converse also holds.

Moreover, if every right R-module is isomorphic to a direct sum of modules in the set S, then R must be right Artinian (20.23). Warfield [72a] showed that a commutative ring can have this property iff it is a principal ideal ring (PIR).

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6 Kaplansky's *Problems in the Theory of Rings*, is indicative of Kaplansky's felicitous influence on these questions (Kaplansky [70]); and, moreover, the second edition of his *Infinite Abelian Groups* contains a prodigious and broad literature commentary on the literature relating directly and indirectly on the first edition (Kaplansky [69]). In a similar vein are Herstein's *Notes from a Ring Theory Conference* (Herstein [71]).

Similarly, Warfield shows in the same paper that every module is a direct sum of indecomposable modules iff R is a PIR.

The theorem of Matlis-Papp (20.5) states that one may decompose injective right modules over R into a direct sum of indecomposable modules iff R is right Noetherian. (What happens if every module is a direct sum of indecomposable modules will be discussed presently.)

Another theorem illustrating the principle that nice properties for the module structure reflect (and are reflected by) nice properties in the ring is a theorem (24.20) which states that every injective right module is projective iff R is quasi-Frobenius (QF) iff every projective right module is injective. For example, every right module can be isomorphic to a direct sum of right ideals only if R is QF, since the condition implies that every injective module is projective. The QF rings are the Artinian rings with a duality between finitely generated right and left modules induced by $\mathrm{Hom}_A(\ ,A)$ for the ring A, and can be characterized as right selfinjective rings with the a.c.c. on left (resp. right) annulets. (See Chapter 24.)

The characterization of when does the category mod-R of *right* R-modules satisfying the property that every module has a projective cover [the dual of the property: every module has an injective hull (Bass [60]; cf. Chapter 22)] is interesting because the characterizing property is the d.c.c. on the principal *left* ideals. This class of rings properly contains the class of left Artinian rings.

Next to the basis theorem for abelian groups, the best known example of the kind of theorems we have been examining is the Wedderburn-Artin theorem (I, 8.9 p. 369) which determines the multiplicative structure of a ring for which every right module is semisimple (=a direct sum of simple right modules): the ring must be similar to (that is, Morita equivalent to) a finite product of fields. (This still holds when semisimple in the statement is replaced by injective (projective).)

Nakayama [39, 40, 41] similarly characterized Artinian rings over which every module is a direct sum of uniserial modules: such a ring is a serial ring in the sense that every principal indecomposable (=prindec) right or left ideal is a uniserial module. Nakayama characterized Artinian serial rings as Artinian rings over which every finitely generated indecomposable module is an epic image of a prindec. (These rings, and also more generally, rings over which every finitely generated right module is a direct sum of cyclic modules (=right σ-cyclic rings), are taken up in Chapter 25.) A serial QF ring has the property that every right module is a direct sum of cyclic right ideals (25.4.17).

A good deal of module theory is aimed at the description of the indecomposable finitely generated modules (at least over right Noetherian rings when every finitely generated module decomposes into a direct sum of indecomposable modules!) Let M be an indecomposable module over a right Noetherian ring R, assume that M is finitely generated, and let $g(M)$ be the least cardinal of any set of generators of M. In general, there exist indecomposable modules M with ever larger $g(M)$. Indeed, by Higman's theorem [54], this happens whenever R is the group algebra in characteristic p with noncyclic p-Sylow subgroup G of finite order n; in particular, finite rings can have this property! (However, in the case of cyclic p-Sylow subgroup, n is a bound on the "number" of indecomposable modules (Kasch-Kneser-Kuppisch [57]).)

Next assume a bound on the $\{g(M)\}$. This is a reasonable finiteness condition which one frequently encounters in classical algebra, for example, as we have seen, it holds over serial rings. Such a ring is said to be right FBG, or bounded module type. A commutative local FBG ring R has linearly ordered ideals (Warfield [70]), illustrating the strength of FBG.

Another kind of finiteness condition that frequently occurs in the theory of finite dimensional algebras and Artinian rings: does right FBG imply finiteness of the isomorphism classes of indecomposable finitely generated right modules? A ring with the latter property is said to be right FFM, or finite module type. (Serial rings are right and left FFM rings.) In this notation the question just stated can be stated as the validity of the implication FBG $\Rightarrow$ FFM. For algebras of finite dimension over a field this was called the Brauer-Thrall conjecture, and was proved by Roiter [68]. For Artinian rings, Auslander [74] proved the conjecture utilizing notably different methods.

Although we have not included these theorems in the text, they are typical of many theorems in the text, in fact, extensions of them, and because of their importance we take this opportunity to acquaint the reader with these results.

Auslander [74, Cor. 4.8] and Ringel and Tachikawa in Tachikawa [73, p. 129, Cor. 9.5] prove: Let R be a right Artinian right FFM ring. Then, every indecomposable right R-module is finitely generated, and every right module is a direct sum of indecomposable modules.

Moreover, Tachikawa [73] also shows that all modules have decompositions which complement direct summands (cds) in the sense if $M=\bigoplus_{i\in I} M_i$ is such a decomposition, then for any direct summand P, there is a subset J of I such that [7] $M=(\bigoplus_{j\in J} M_j)\oplus P$. Fuller-Reiten prove a converse for rings over which right and left modules have decompositions which cds. Auslander [74] showed that Artin algebras are FFM provided only that every indecomposable left module is finitely generated. Moreover, a theorem of Faith-Walker [67] puts on the finishing touch: if every injective left module is a direct sum of finitely generated modules, then R is left Artinian 20.17. (This property characterizes commutative Artinian rings 20.18: as stated earlier, if every left module decomposes into a direct sum of modules of bounded cardinality, then R is left Artinian 20.23.)

To return to cyclic modules: why are they so important to many structure theories? A possible answer: every right FBG ring R is similar to a ring A over which every generated module is a direct sum of cyclic modules. (This is trivial to prove: see 20.39.) Moreover, in this case, R is right FFM iff A has at most finitely many nonisomorphic indecomposable cyclic modules. Since isomorphic modules have the same annihilating ideal, in some cases, for example, when R is right Artinian, then right FFM implies that the lattice of ideals is finite 20.4.4. This notwithstanding, the right ideal structure of a right FFM ring has yet to be determined that would make the theory comparable to that for serial rings, and appears to be a problem to which a solution will have a reasonable expectation of clearing out a jungle of present-day special cases.

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[7] This concept of Anderson-Fuller [72], and its relationship to ideas of Crawley-Jonsson [64] and Warfield [72b], is discussed in Notes for Chapter 21.

III. Algebra

I will close the Preface with a few generalities and some specifics. Algebra, like other branches of mathematics, systematically exploits quite general geometric properties—I am thinking of simple things like up-down, left-right, co-ordinations, sequences, symmetries-asymmetries, subdivisions, partitions, the "pigeon-hole" principle, equivalence, dualities, and the like. (To continue the list would make omissions appear more ominous than I intend!)

Because of the generality which mathematical statements are capable of, the term "abstract" is often applied to what in reality is quite specific. For example, theorems are published, but "theories" rarely, if at all. (The German use of the word *Satz*, or sentence, for theorem illustrates this point nicely, I think.)

The confusion between what is abstract and what is concrete arises, I believe, from the mathematician's passion for making the concrete as general as possible, by eliminating unnecessary, that is, unused, hypotheses from the statements. But it is, first of all, and above all, the concrete, the real, and indeed the useful, that involves the mathematician. (I do not mean to exclude beauty—the beauty of mathematical statements is a useful organizing principle for the sensitive mind.) The ethic is to eliminate waste, or the wasted, to determine the real, by making the vague, or imprecise, meaningful (if possible!).

Let me illustrate this with an example: G. Köthe proved that an Artinian commutative ring R with the property

(right $\sum$-cyclic) every right module is a direct sum of cyclics

is a uniserial (einreihig) ring. Cohen and Kaplansky [51] countered with the observation that it was redundant to assume that R is Artinian. S.U. Chase [60], then a student of Kaplansky, proved commutativity is not necessary to assert the ring is right Artinian, and moreover, that finitely generated modules can replace the cyclics in the statement. (But, then, the ring is no longer necessarily serial, of course.) Finally, it was noticed that finite cardinality of the modules in the direct summands played no role; if there exists a *set* of modules such that every right module decomposes into a direct sum of modules isomorphic to modules in that set, then the ring is right Artinian. The proof of this, given in Chapter 20, makes heavy use of another theorem of Chase [60] on direct sum decompositions of modules: If there is a cardinal number c not less than the cardinal of R such that the product R^c is a pure submodule (for example, a direct summand) of a direct sum of right R-modules having cardinal not exceeding c, then R satisfies the d.c.c. on principal left ideals. These latter rings are in fact rings which Bass [60] (then another Kaplansky student!) studied in the connection with the requirement that all right modules have projective covers. Bass called these rings right perfect rings, and much of the structure theory of Artinian nonsemisimple rings was extended by Bass to perfect rings. (An account of this is given in Chapter 22.)

The complete structure of $\sum$-cyclic rings, a problem posed by Köthe [35] is still unknown. Nakayama's papers on (generalized uni) serial rings (Nakayama [39, 40, 41]) showed the rings to be more general than serial rings. Kawada gave an exhaustive study, and complete solution in a very special case, but even so there were 19 (or so) formidable conditions deemed necessary and sufficient.

In turn, Kaplansky [69] asked for the structure of right σ-cyclic rings, or those over which every finitely generated module is a direct sum of cyclic modules.

As noted, Köthe's theorem solved the problem for commutative $\sum$-cyclic rings, and while Kaplansky [49, 52] *et al.* solve the problem for commutative local rings, it still remains open for arbitrary commutative rings. (Some of these matters are taken up in Chapters 20 and 25.)

In the meantime, research continues on the problems treated here, and the related problems on the right ideal structure of rings of finite module type (FFM rings) discussed in the Introduction to Volume II.

Having succeeded, at the very least, in connecting the first two parts of the preface, and having already described a number of ideas from Nakayama, let us remember his closing remark to the International Congress of Mathematicians, Amsterdam, 1950 (reprinted in Nakayama [50b]) on a closely related subject:

It seems to the writer that our topics possess a somewhat deeper connection with each other than was said in the beginning.

IV. Principal Contributors

The principal contributors to the contents are: S.A. Amitsur, E. Artin, K. Asano, M. Auslander, G. Azumaya, R. Baer, H. Bass, J.A. Beachy, J.E. Björk, R. Brauer, G. Burnside, S.U. Chase, A.W. Chatters, C. Chevalley, I.S. Cohen, P.M. Cohn, I. Connell, R. Croisot, J.H. Cozzens, R. Dedekind, J.A. Dieudonné, B. Eckmann, S. Eilenberg, D. Eisenbud, G.D. Findlay, H. Fitting, G. Frobenius, E. Feller, K.R. Fuller, C.F. Gauss, D.T. Gill, A.W. Goldie, K.R. Goodearl, P. Griffith, M. Harada, I.N. Herstein, O. Hölder, D. Hilbert, C. Hopkins, M. Ikeda, N. Jacobson, R.E. Johnson, I. Kaplansky, E. Kolchin, G. Köthe, L. Kronecker, W. Krull, J.P. Lafon, C. Lanski, J. Lambek, L. Lesieur, J. Levitzki, L.S. Levy, E. Maschke, E. Matlis, N.H. McCoy, Y. Miyashita, K. Morita, T. Nakayama, E. Noether, O. Ore, B.L. Osofsky, Z. Papp, S. Perlis, R. Remak, J.C. Robson, F.L. Sandomierski, O. Schmidt, A. Schopf, O. Schreier, I. Schur, R. Shock, L. Small, E. Steinitz, R. Swan, E. Swokowski, H. Tachikawa, Y. Utumi, P. Vámos, J. von Neumann, E.A. Walker, R.B. Warfield, Jr., D.B. Webber, E.T. Wong, and J.H.M. Wedderburn.

Readers familiar with my research interests will not be surprised to see to what extent the text is a delineation of the dominant roles that injective and projective modules have played in the simplification, clarification, extension, and deepening of much of classical algebra. It would be pointless to adduce specific examples here, since so much of the text is devoted to such examples, but even the Chevalley-Jacobson density theorem fits into this framework!

It goes without saying that another author would have made other choices for inclusion in the text; but some papers, especially "break-throughs" like Roiter [68], touch on a number of important theorems of classical mathematics, and therefore invite a *rewriting* of mathematics. To relegate papers of such power to the status of an "inclusion" would be a mutilation not only of the potential of such a paper to *revise* mathematics, but also of what I had planned to do. *Of course*, mathematics has not stood still!

V. Acknowledgements

Albert Einstein has been quoted as saying that teachers should set an example for their students—of what to avoid if they cannot be the other kind. In this context

I must confess: I began this book in Summer 1965 at the Institute for Advanced Study, continued it at Berkeley in 1965–1966 (where I finished the prototype of Volume II in Summer 1966), made revisions and additions, notably category theory (described in the Introduction to Volume I), in Princeton in 1969 (both years provided for by Faculty Fellowships awarded by the Rutgers Research Council), and I am writing this in my sabbatical year at the Institute for Advanced Study, where I began. *Sic semper scriptor!*

Mere mention of the splendid faculty and facilities of the Institute for Advanced Study would not sufficiently convey their importance to my work. The dedication to mathematics and rational thought of those who study there provided me with an unending source of inspiration, and an inexpressible joy of being, which sustains me in all my work.

More than that, there is a freedom of inquiry and thought that is powerfully unique even among other truly fine institutions. May the present faculty accept this small tribute as a sign of may admiration for, and involvement with, them in their constant struggle to preserve this fierce intellectual freedom.

Without the understanding interest of the editors of the Springer-Verlag, this book might never had been printed. There are enormous expenses involved in the typesetting required for the fine Springer books, and a comittment to publish is not a lighthearted one to make. Most of all I am indebted to one of the two Chief Editors of the Grundlehren volumes, Professor Beno Eckmann, for making this book possible. I also offer my grateful thanks to Professor Albrecht Dold for his part in making this decision.

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Contents

* Because of the increase in size (in revision) of Part V, Part VI (Commutative Rings, Hereditary Rings, Separable Algebras and the Brauer Group), comprising of Chapters 27–32, will not appear as part of Volume II as announced in Volume I. Thus, Chapters 17–26 comprise all of Volume II.

Special Symbols and Terms

These are listed in Volume I, p. XXIII, but the list does not contain the conventions: mod-R denotes the category or class of all right R-modules, for a ring R, and R-mod is the left-right symmetry. Unless specified otherwise, a ring R will have an identity element 1, and mod-R denotes the category of unital modules in the sense that for every M of mod-R, $x1 = x \; \forall \; x \in M$. Usually, homomorphisms are written on the side opposite scalars, as discussed in Volume I, pp. 119–120.

A non-standard term used throughout is the word similar, applied to two rings A and B, to denote an equivalence mod-$A \approx$ mod-B of categories. (In the literature, the expression A is Morita equivalent to B is used.) Similarity is taken up in Volume I on p. 217, The Morita Theorem, 4.29.[1] The notation $A \sim B$ denotes the similarity relation, and it is reflexive, symmetric, and transitive.

We employ a now standard symbol: $A \hookrightarrow B$ indicates an embedding of a group or module A in B. However, in some places the printers have substituted a symbol $C \rightsquigarrow D$ to indicate a functor, replacing the symbol used in Volume I!

The symbol ring-1 indicates a ring in which an identity element is not assumed. (Thus, any proper ideal is a ring-1.)

As in Volume I, I have found it convenient to quote relevant literature in the form of "exercises", and I wish to emphasize that this is indeed a convenience, as well as a stimulation to the imagination of the intending reader, and in no way is to be interpreted as a relegation to exercises some very important theorems. (Many of results of mine and coauthors are found thus!) It is unlikely that many of the proofs of these will be discovered by the neophyte, yet I do believe this is the way mathematics ought to be learned: *do or die!* [As compensation many papers are published!!]

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[1] Ordinarily in the text I will abbreviate such a reference by (I, 4.29, p. 217).