



THE NUMBER π

Pierre Eymard

Jean-Pierre Lafon

 **AMS**
AMERICAN MATHEMATICAL SOCIETY

Table of Contents

Preface	
Ch. 1 Measurement of the Circle	1
1.1 Archimedes' duplication method	1
1.2 Some surfaces and volumes linked to the number π	9
1.3 Gauss's circle problem	27
1.4 Buffon's needle	34
Ch. 2 Wallis's Formula and Some Others	43
2.1 Viete's infinite product	44
2.2 General notes on infinite products	46
2.3 Wallis's infinite product	48
2.4 $\sqrt{\pi}$ and the game of heads or tails	50
2.5 Stirling's equivalent	52
2.6 The formula of Gregory and Leibniz	53
2.7 A series for the inverse tangent due to Euler	55
2.8 Other series and integrals to express π	56
2.9 Formulae of Machin type	58
2.10 On the equation $\pi/4 = \arctan 1/n_1 + \dots + \arctan 1/n_p$	62
2.11 Continued fractions for π	65
2.12 The formula of Bailey, Borwein and Plouffe	79
Ch. 3 Euler, Euler Again, Always Euler	83
3.1 The place of π and of e in an analysis course	83
3.2 π and Fourier series	90
3.3 Some proofs of the formula [actual symbol not reproducible]	93
3.4 The dilogarithm function	97
3.5 Asymptotic behaviour of the mean of certain arithmetic functions	99
3.6 The Bernoulli numbers in analysis	106
3.7 The Gamma function	120
Ch. 4 Squaring the Circle	129
4.1 Ruler and compass constructions	129
4.2 Questions of irrationality	134
4.3 Lambert's proof of irrationality	137
4.4 A proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ due to F. Beukers	143
4.5 Transcendental numbers	146
4.6 Algebraic characterization of constructible numbers	153
4.7 The transcendency of e and of π	168
4.8 A measure of irrationality for π	177
4.9 Regular polygons	182
Ch. 5 π and Elliptic Integrals	195
5.1 Elliptic integrals and lemniscate arcs	197
5.2 The arithmetic-geometric mean	205
5.3 Algorithms for calculating integrals of the first and second kind	208
5.4 An algorithm for calculating the digits of π due to the brothers Borwein	219
5.5 Abel and division of the lemniscates	221
5.6 From Jacobi to Ramanujan	237
5.7 π and Ramanujan	243
5.8 An overview of the proof by J. Borwein and P. Borwein	251
Ch. 6 Solutions to the Exercises	255
Bibliography	313
Index	319