

Editorial Board

S. Axler F.W. Gehring P.R. Halmos

Springer

New York

Berlin

Heidelberg

Barcelona

Budapest

Hong Kong

London

Milan

Paris

Santa Clara

Singapore

Tokyo

Graduate Texts in Mathematics

- 1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
- 2 OXTOBY. Measure and Category. 2nd ed.
- 3 SCHAEFER. Topological Vector Spaces.
- 4 HILTON/STAMMBACH. A Course in Homological Algebra.
- 5 MAC LANE. Categories for the Working Mathematician.
- 6 HUGHES/PIPER. Projective Planes.
- 7 SERRE. A Course in Arithmetic.
- 8 TAKEUTI/ZARING. Axiomatic Set Theory.
- 9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
- 10 COHEN. A Course in Simple Homotopy Theory.
- 11 CONWAY. Functions of One Complex Variable I. 2nd ed.
- 12 BEALS. Advanced Mathematical Analysis.
- 13 ANDERSON/FULLER. Rings and Categories of Modules. 2nd ed.
- 14 GOLUBITSKY/GUILLEMIN. Stable Mappings and Their Singularities.
- 15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
- 16 WINTER. The Structure of Fields.
- 17 ROSENBLATT. Random Processes. 2nd ed.
- 18 HALMOS. Measure Theory.
- 19 HALMOS. A Hilbert Space Problem Book. 2nd ed.
- 20 HUSEMOLLER. Fibre Bundles. 3rd ed.
- 21 HUMPHREYS. Linear Algebraic Groups.
- 22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
- 23 GREUB. Linear Algebra. 4th ed.
- 24 HOLMES. Geometric Functional Analysis and Its Applications.
- 25 HEWITT/STROMBERG. Real and Abstract Analysis.
- 26 MANES. Algebraic Theories.
- 27 KELLEY. General Topology.
- 28 ZARISKI/SAMUEL. Commutative Algebra. Vol.I.
- 29 ZARISKI/SAMUEL. Commutative Algebra. Vol.II.
- 30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
- 31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
- 32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
- 33 HIRSCH. Differential Topology.
- 34 SPITZER. Principles of Random Walk. 2nd ed.
- 35 WERMER. Banach Algebras and Several Complex Variables. 2nd ed.
- 36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
- 37 MONK. Mathematical Logic.
- 38 GRAUERT/FRITZSCHE. Several Complex Variables.
- 39 ARVESON. An Invitation to C^* -Algebras.
- 40 KEMENY/SNELI/KNAPP. Denumerable Markov Chains. 2nd ed.
- 41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory. 2nd ed.
- 42 SERRE. Linear Representations of Finite Groups.
- 43 GILLMAN/JERISON. Rings of Continuous Functions.
- 44 KENDIG. Elementary Algebraic Geometry.
- 45 LOÈVE. Probability Theory I. 4th ed.
- 46 LOÈVE. Probability Theory II. 4th ed.
- 47 MOISE. Geometric Topology in Dimensions 2 and 3.
- 48 SACHS/WU. General Relativity for Mathematicians.
- 49 GRUENBERG/WEIR. Linear Geometry. 2nd ed.
- 50 EDWARDS. Fermat's Last Theorem.
- 51 KLINGENBERG. A Course in Differential Geometry.
- 52 HARTSHORNE. Algebraic Geometry.
- 53 MANIN. A Course in Mathematical Logic.
- 54 GRAVER/WATKINS. Combinatorics with Emphasis on the Theory of Graphs.
- 55 BROWN/PEARCY. Introduction to Operator Theory I: Elements of Functional Analysis.
- 56 MASSEY. Algebraic Topology: An Introduction.
- 57 CROWELL/FOX. Introduction to Knot Theory.
- 58 KOBLITZ. p -adic Numbers, p -adic Analysis, and Zeta-Functions. 2nd ed.
- 59 LANG. Cyclotomic Fields.
- 60 ARNOLD. Mathematical Methods in Classical Mechanics. 2nd ed.

continued after index

Günter Ewald

Combinatorial Convexity and Algebraic Geometry

With 130 Illustrations



Springer

Günter Ewald
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstrasse 150
D-44780 Bochum
Germany

Editorial Board

S. Axler
Department of
Mathematics
Michigan State University
East Lansing, MI 48824
USA

F.W. Gehring
Department of
Mathematics
University of Michigan
Ann Arbor, MI 48109
USA

P.R. Halmos
Department of
Mathematics
Santa Clara University
Santa Clara, CA 95053
USA

Mathematics Subject Classification (1991): 52-01, 14-01

Ewald, Günter, 1929–

Combinatorial convexity and algebraic geometry / Günter Ewald.

p. cm.—(Graduate texts in mathematics; 168)

Includes bibliographical references and index.

ISBN-13: 978-1-4612-8476-5 e-ISBN-13: 978-1-4612-4044-0

DOI: 10.1007/978-1-4612-4044-0

1. Combinatorial geometry. 2. Toric varieties. 3. Geometry.

Algebraic. I. Title. II. Series.

QA639.5.E93 1996

516'.08—dc20

96-11792

Printed on acid-free paper.

© 1996 Springer-Verlag New York, Inc.

Softcover reprint of the hardcover 1st edition 1996

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

Production managed by Lesley Poliner; manufacturing supervised by Johanna Tschebull.

Photocomposed pages prepared from the author's \TeX files.

9 8 7 6 5 4 3 2 1

*To Hanna
and our children
Daniel, Sarah, Anna, Esther, David*

Preface

The aim of this book is to provide an introduction for students and nonspecialists to a fascinating relation between combinatorial geometry and algebraic geometry, as it has developed during the last two decades. This relation is known as the theory of toric varieties or sometimes as torus embeddings.

Chapters I–IV provide a self-contained introduction to the theory of convex polytopes and polyhedral sets and can be used independently of any applications to algebraic geometry. Chapter V forms a link between the first and second part of the book. Though its material belongs to combinatorial convexity, its definitions and theorems are motivated by toric varieties. Often they simply translate algebraic geometric facts into combinatorial language. Chapters VI–VIII introduce toric varieties in an elementary way, but one which may not, for specialists, be the most elegant.

In considering toric varieties, many of the general notions of algebraic geometry occur and they can be dealt with in a concrete way. Therefore, Part 2 of the book may also serve as an introduction to algebraic geometry and preparation for farther reaching texts about this field.

The prerequisites for both parts of the book are standard facts in linear algebra (including some facts on rings and fields) and calculus. Assuming those, all proofs in Chapters I–VII are complete with one exception (IV, Theorem 5.1). In Chapter VIII we use a few additional prerequisites with references from appropriate texts.

The book covers material for a one year graduate course. For shorter courses with emphasis on algebraic geometry, it is possible to start with Part 2 and use Part 1 as references for combinatorial geometry.

For each section of Chapters I–VIII, there is an addendum in the appendix of the book. In order to avoid interruptions and to minimize frustration for the beginner, comments, historical notes, suggestions for further reading, additional exercises, and, in some cases, research problems are collected in the Appendix.

Acknowledgments

This text is based on lectures I gave several times at Bochum University. Many colleagues and students have contributed to it in one way or another.

There are seven people to whom I owe special thanks. Jerzy Jurkiewicz (Warsaw) gave me much advice and help in an early stage of the writing. Gottfried Barthel and Ludger Kaup (Konstanz) thoroughly analyzed and corrected large parts of the first six chapters, and even rewrote some of the sections. In a later stage, Jaroslav Włodarczyk (Warsaw) worked out strong improvements of Chapters VI and VII. Robert J. Koelman prepared the illustrations by computer. Finally, Bernd Kind suggested many changes and in addition has supervised the production of the text and patiently solved all arising technical problems.

Also Markus Eikelberg, Rolf Gärtner, Ralph Lehmann, and Uwe Wessels made important contributions. Michel Brion, Dimitrios Dais, Bernard Teissier, Günter Ziegler added remarks, and Hassan Azad, Katalin Bencsath, Peter Braß, Sharon Castillo, Reinhold Matmann, David Morgan, and Heinke Wagner made corrections to the text. Elke Lau and Elfriede Rahn did the word processing of the computer text.

I thank all who helped me, in particular, those who are not mentioned by name.

Günter Ewald

Contents

| | |
|--------------|------|
| Preface | vii |
| Introduction | xiii |

Part 1

Combinatorial Convexity

| | |
|--|----|
| I. Convex Bodies | 3 |
| 1. Convex sets | 3 |
| 2. Theorems of Radon and Carathéodory | 8 |
| 3. Nearest point map and supporting hyperplanes | 11 |
| 4. Faces and normal cones | 14 |
| 5. Support function and distance function | 18 |
| 6. Polar bodies | 24 |
| II. Combinatorial theory of polytopes and polyhedral sets | 29 |
| 1. The boundary complex of a polyhedral set | 29 |
| 2. Polar polytopes and quotient polytopes | 35 |
| 3. Special types of polytopes | 40 |
| 4. Linear transforms and Gale transforms | 45 |
| 5. Matrix representation of transforms | 53 |
| 6. Classification of polytopes | 58 |
| III. Polyhedral spheres | 65 |
| 1. Cell complexes | 65 |
| 2. Stellar operations | 70 |
| 3. The Euler and the Dehn–Sommerville equations | 78 |
| 4. Schlegel diagrams, n -diagrams, and polytopality of spheres | 84 |
| 5. Embedding problems | 88 |
| 6. Shellings | 92 |

| | | |
|------------|---|------------|
| 7. | Upper bound theorem | 96 |
| IV. | Minkowski sum and mixed volume | 103 |
| 1. | Minkowski sum | 103 |
| 2. | Hausdorff metric | 107 |
| 3. | Volume and mixed volume | 115 |
| 4. | Further properties of mixed volumes | 120 |
| 5. | Alexandrov–Fenchel’s inequality | 129 |
| 6. | Ehrhart’s theorem | 135 |
| 7. | Zonotopes and arrangements of hyperplanes | 138 |
| V. | Lattice polytopes and fans | 143 |
| 1. | Lattice cones | 143 |
| 2. | Dual cones and quotient cones | 148 |
| 3. | Monoids | 154 |
| 4. | Fans | 158 |
| 5. | The combinatorial Picard group | 167 |
| 6. | Regular stellar operations | 179 |
| 7. | Classification problems | 186 |
| 8. | Fano polytopes | 192 |

Part 2

Algebraic Geometry

| | | |
|-------------|---|------------|
| VI. | Toric varieties | 199 |
| 1. | Ideals and affine algebraic sets | 199 |
| 2. | Affine toric varieties | 214 |
| 3. | Toric varieties | 224 |
| 4. | Invariant toric subvarieties | 234 |
| 5. | The torus action | 238 |
| 6. | Toric morphisms and fibrations | 242 |
| 7. | Blowups and blowdowns | 248 |
| 8. | Resolution of singularities | 252 |
| 9. | Completeness and compactness | 257 |
| VII. | Sheaves and projective toric varieties | 259 |
| 1. | Sheaves and divisors | 259 |
| 2. | Invertible sheaves and the Picard group | 267 |
| 3. | Projective toric varieties | 273 |
| 4. | Support functions and line bundles | 281 |
| 5. | Chow ring | 287 |
| 6. | Intersection numbers. Hodge inequality | 290 |

| | |
|--|------------|
| 7. Moment map and Morse function | 296 |
| 8. Classification theorems. Toric Fano varieties | 303 |
| VIII. Cohomology of toric varieties | 307 |
| 1. Basic concepts | 307 |
| 2. Cohomology ring of a toric variety | 314 |
| 3. Čech cohomology | 317 |
| 4. Cohomology of invertible sheaves | 320 |
| 5. The Riemann–Roch–Hirzebruch theorem | 324 |
| Summary: A Dictionary | 329 |
| Appendix | |
| Comments, historical notes, further exercises, research problems, suggestions for further reading | 331 |
| References | 343 |
| List of Symbols | 359 |
| Index | 363 |

Introduction

Studying the complex zeros of a polynomial in several variables reveals that there are properties which depend not on the specific values of the coefficients but only on their being nonzero. They depend on the exponent vectors showing up in the polynomial or, more precisely, on the lattice polytope which is the convex hull of such vectors. This had already been discovered by Newton and was taken into consideration by Minding and some other mathematicians in the nineteenth century. However, it had practically been forgotten until its rediscovery around 1970, when Demazure, Oda, Mumford, and others developed the theory of toric varieties.

The starting point lay in algebraic groups. Properties of zeros of polynomials that depend only on the exponent vectors do not change if each coordinate of any solution is multiplied by a nonvanishing constant. Such transformations are effected by diagonal matrices with nonzero determinants. They form a group which can be represented by \mathbb{C}^{*n} where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is the multiplicative group of complex numbers. \mathbb{C}^{*n} (for $n = 2$ having, topologically, an ordinary torus as a retract) is called an algebraic torus. Demazure succeeded in combinatorially characterizing those regular algebraic varieties on which a torus operates with an open orbit. Oda, Mumford, and others extended this to the nonregular case and termed the introduced varieties torus embeddings or toric varieties.

Once the combinatorial characterization had been achieved, it gave way to defining toric varieties without starting from algebraic groups by use of combinatorial concepts like lattice cones and the algebras defined by monoids of all lattice points in cones. This is the path we follow in the present book.

Toric varieties—being a class of relatively concrete algebraic varieties—may appear to relate combinatorics to old-fashioned, say, up to 1950, algebraic geometry. This is not the case. Actually, the more recent way of thought provides the tools for building a wide bridge between combinatorial and algebraic geometry. Notions like sheaves, blowups, or the use of homology in algebraic geometry are such tools.

In the first part of the book, we have naturally limited the topics to those which are needed in the second part. However, there was not much to be omitted. Coming

from combinatorial convexity, it is quite a surprise how many of the traditional notions like support function or mixed volume now appear in a new light.

In our attempt to present a compact introduction to the theory of convex polytopes, we have sought short proofs. Also, a coordinate-free approach to Gale transforms seemed to fit particularly well into the needs of later applications. Similarly, in Part 2 we spent much energy on simplifications. Our definition of intersection numbers and a discussion of the Hodge inequality working without the tools of algebraic topology are some of the consequences.

A natural question concerning the relationship between combinatorial and algebraic geometry is “Does the algebraic geometric side benefit more from the combinatorial side than the combinatorial side does from the algebraic geometric one?” In this text the former is true. We prove algebraic geometric theorems from combinatorial geometric facts, “turning around” the methods often applied in the literature. There is only one exception in the very last section of the book. We quote a toric version of the Riemann–Roch–Hirzebruch theorem without proof and draw combinatorial conclusions from it. A purely combinatorial version of the theorem due to Morelli [1993a] would require more work on so-called polytope algebra.

Many related topics have been omitted, for example, matroid theory or the theory of Stanley–Reisner rings and their powerful combinatorial implications. The reader familiar with such topics may recognize their links to those covered here and detect the common spirit of mathematical development in all of them.