

# Lecture Notes in Mathematics

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# Orthomorphism Graphs of Groups

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## Preface

The study of orthomorphism graphs of groups has its origin in the use of orthomorphisms and complete mappings to construct mutually orthogonal sets of Latin squares. To help other mathematicians who wish to work in this area, a reference work is needed. None exists at present and work on the subject is scattered throughout the literature, often in a form that does not suggest any connection to orthomorphisms.

In writing this monograph I have tried to do more than survey work done so far. In this monograph I have attempted to consolidate known results and applications, to create a unified body of knowledge and to provide other mathematicians with the tools needed to work in this area. I have tried to lay down the beginnings of a framework for the theory of orthomorphism graphs of groups and their applications, incorporating topics from algebra and geometry into this theory. As one of the hopes for this project was that it would stimulate research in this area, I have suggested many problems and directions for future research in this field, which should provide algebraists and geometers as well as other researchers in combinatorics with questions to work on.

The material in this book should be accessible to any graduate student who has taken courses in group theory and field theory.

I would like to thank Wright State University for its support during the writing of this manuscript, part of which was written while on sabbatical, and my colleagues Manley Perkel, who worked with me in generating orthomorphisms using Cayley, and Terry McKee, who read part of the manuscript, and the referees for their helpful suggestions.

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